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On Cyclic Subspaces and Unicellularity of the Operator $(Vf)(x) = q(x)\int_0^x w(t)f(t)dt$

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(*Presented by A. E. Shishkov*)

Abstract. Let $q(x) \in L_p[0,1]$, let $w(x) \in L_{p'}[0,1]$, and let $\overline{q(x)w(x)} = q(x)w(x) \neq 0$ for almost all $x \in [0,1]$. We describe the cyclic subspaces, spectral multiplicity and disc-characteristic of the operator $(V_{q,w}f)(x) = q(x)\int_0^x f(t)w(t)dt$ and its natural powers in the space $L_p[0,1]$. It is also shown that the unicellularity of the operator $V_{q,w}^n$ is equivalent to its quasisimilarity to the operator $cJ^n := cV_{1,1}^n$ and to the condition $\operatorname{sign} q(x)w(x) = const$ almost everywhere on [0,1].

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1. Introduction

It is well known [3, 17] that the operator of integration $J : f(x) \to \int_0^x f(t)dt$ is unicellular in the spaces $L_p[0,1]$ for $p \in [1,\infty)$ and the lattice of its invariant subspaces is antiisomorphic to the segment $[0,1]$. This statement remains true (see [3, 17]) for the positive powers of the operator of integration:

$$J^\alpha : f(x) \to \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt, \qquad \alpha > 0.$$

Moreover, the corresponding lattices of invariant and hyperinvariant subspaces have the form (see [1, 3, 17])

$$\operatorname{Lat} J^\alpha = \operatorname{Hyplat} J^\alpha = \{E_a := \chi_{[a,1]} L_p[0,1] : 0 \leqslant a \leqslant 1\}. \tag{1.1}$$

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The description (1.1) of the lattice $\operatorname{Lat} J^\alpha$ immediately yields the following description of the set $\operatorname{Cyc} J^\alpha$ of cyclic vectors of the operator J^α:

$$f \in \operatorname{Cyc} J^\alpha \Leftrightarrow \int_0^\varepsilon |f(x)|^p \, dx > 0, \qquad \varepsilon > 0. \tag{1.2}$$

Condition (1.2) is called the ε-condition.

In [12, 13], Malamud performed the detailed investigation of the operator $A = J^\alpha \otimes B$ acting in the space $L_p[0,1] \otimes \mathbb{C}^n$ as the tensor product of the operator J^α and an arbitrary nonsingular diagonal $n \times n$ matrix $B = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Thus, in [12, 13], he determined the spectral multiplicity μ_A of this operator and described the lattices $\operatorname{Lat} A$ and $\operatorname{Hyplat} A$ of its invariant and hyperinvariant subspaces and the set $\operatorname{Cyc} A$ of its cyclic subspaces. The description of the set $\operatorname{Cyc} A$ proposed in [12, 13] was somewhat unexpected. Indeed, it was obtained in terms of $*$-ranks and $*$-determinants (see Definition 3.1) constructed from the components of vectors generating a cyclic subspace. Earlier, the notion of $*$-determinant appeared in [11, 15] in a different case.

In the present work, we mainly study the operator

$$V_{q,w} : f(x) \to q(x) \int_0^x f(t) w(t) \, dt \tag{1.3}$$

and its natural powers. For $q \equiv w \equiv 1$, the operator $V_{q,w}$ coincides with the operator of integration.

In [8], Joo Ho Kang studied the operator $V_{q,w}$ in $L_2[0,1]$ over the field of real numbers in the case where the functions $q(x)$ and $w(x)$ are positive and continuous. Under these conditions imposed on the functions q and w, he proved the unicellularity of the operator $V_{q,w}$ and described the set $\operatorname{Cyc} V_{q,w}$ of its cyclic vectors.

Note that, for positive and absolutely continuous $q, w \in AC[0,1]$, the fact of similarity of the operator $V_{q,w}$ to the operator cJ and, hence, its unicellularity, follows from the general result obtained by Malamud in [14] concerning the similarity of the operator K $((Kf)(x) = \int_0^x k(x,t) f(t) dt)$ to the operator of integration J.

Note that operators of the form (1.3) are of interest in the theory of Brownian motion. Thus, recently, the asymptotics of approximative and entropy numbers of the operator $V_{q,w} : L_p[0,\infty) \longrightarrow L_q[0,\infty)$ have been established in [10]. In the same work, one can also find the analysis of the possibility of application of these results to the case of weighted Wiener processes.

In the present work, we study the powers $V_{q,w}^n$ of an operator of the form (1.3) without imposing the condition $q(x)w(x) > 0$ and the condition of continuity of q and w. At the same time, we assume that the function $q(x)w(x)$ is real.

In what follows, we deduce necessary and sufficient conditions for q and w under which the operator $V_{q,w}$ is not only unicellular but also similar to the operator cJ. It turns out that the unicellularity of the operator $V_{q,w}^n$ is equivalent to its quasisimilarity to the operator of integration and to the requirement that $q(x)w(x) > 0$ almost everywhere on [0,1].

Most of the results are also new for the case when $q(x)w(x) > 0$.

However, the case of an alternating function $q(x)w(x)$ appears to be much more interesting. We now list several new effects established in what follows:

(1) the operator $V_{q,w}$ loses the property of unicellularity;

(2) its restrictions to some invariant subspaces are quasisimilar to operators of the form $A = J \otimes B$;

(3) under certain conditions imposed on the functions q and w, the operator $V_{q,w}$ can be cyclic but not unicellular.

As one of the main results obtained in the present work, we can mention the determination of the spectral multiplicity of powers of the operator $V_{q,w}$ and the description of the set of their cyclic subspaces. As in [12, 13], the indicated description is presented in terms of $*$-ranks and $*$-determinants of functional matrices constructed for a system of vectors generating a cyclic subspace. Thus, if $q(x) = \chi_{[0,a]}(x) - \chi_{[a,1]}(x)$, then $\mu_{V_{q,1}^2} = 2$ and the subspace $E := \operatorname{span}\{f_1, f_2\}$ is cyclic for the operator $V_{q,1}^2$ if only if

$$*\text{-rank}\begin{pmatrix} f_1(x) & f_1(a-x) + f_1(a+x) \\ f_2(x) & f_2(a-x) + f_2(a+x) \end{pmatrix} = 2$$

(see also Examples 3.1 and 3.2).

In this connection, it is worth noting that the concept of spectral multiplicity plays an important role in the theory of control [16]. Indeed, a linear dynamical system is described by the equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad t \geq 0 \tag{1.4}$$

with operators $A \in [X]$ and $B = B \upharpoonright U$, $B \in [U, X]$, where X and U are Banach spaces (the space of states and the control subspace, respectively), $x(t) \in X$, and $u(t) \in U$.

The first problem of the theory of control is to find all spaces U satisfying the following condition of controllability: for any state $x \in X$, any $\varepsilon > 0$, and any time $t > 0$, there exists a (smooth) control function $u(\cdot)$ for which the solution of system (1.4) appears in the ε-neighborhood of x, $\|x - x(t)\| < \varepsilon$, starting from a certain fixed initial state $x(0)$ (e.g., $x(0) = 0$).

The well-known Kalman theorem states that the system A, $B \restriction U$ is controllable if and only if $BU \in \operatorname{Cyc} A$, i.e., in the case where the minimum dimensionality of the control subspace in (1.4) is equal to μ_A.

In analyzing some other problems of the theory of control, Nikol'skii and Vasyunin [16] introduced a new characteristic of the operator A, namely, its disc-characteristic (see Definition 2.3) specifying the possibility of decreasing the dimensionality of the control subspace for system (1.4) without losing the property of controllability.

We determine the disc-characteristic of powers of the operator $V_{q,w}$ and show that this characteristic coincides with the spectral multiplicity of the operator.

Notation.

(1) X_1 and X_2 are Banach spaces, (2) $[X_1, X_2]$ is the space of linear bounded operators from X_1 to X_2, $[X] := [X, X]$, (3) $\operatorname{Lat} T$ is the lattice of invariant subspaces of the operator $T \in [X]$, (4) $\ker T = \{x \in X : Tx = 0\}$ is the kernel of the operator T, (5) $\mathfrak{R}(T) = \{Tx : x \in X\}$ is the range of the operator $T \in [X]$, (6) $\operatorname{span} E$ is the closed linear span of the set $E (\subset X)$ in X, (7) $\operatorname{supp} f$ is the support of the function $f(x)$, and (8) $r * f$ denotes the convolution of functions $r,\ f \in L_1[0,1] : r * f := \int_0^x r(x-t) f(t) dt$.

All results obtained in the present work were announced in [5, 6].

2. Quasisimilarity of the Operator $V_{q,w}^{\alpha}$ to the Operator $V_{r,1}^{\alpha}$

Lemma 2.1. *Let $q \in L_p[0,1]$, $w \in L_{p'}[0,1]$ ($p'^{-1} + p^{-1} = 1$), and $q(x)w(x) = \overline{q(x)w(x)} \neq 0$ for almost all $x \in [0,1]$. Then the operator*

$$V_{q,w} : f \to q(x) \int_0^x f(t) w(t) dt$$

acting in the space $L_p[0,1]$ is a Volterra operator and $\operatorname{Ker} V_{q,w} = \{0\}$.

Proof. The boundedness of the operator $V_{q,w}$ and the equality $\operatorname{Ker} V_{q,w} = \{0\}$ are obvious. Since $q \in L_p[0,1]$ and $w \in L_{p'}[0,1]$, by virtue of the Hille–Tamarkin theorem [18], the operator $V_{q,w}$ is compact. In view of the fact that the operator $V_{q,w}$ is compact, its spectral radius is equal to zero [7] and, hence, $V_{q,w}$ is a Volterra operator. $\square$

Proposition 2.1. *Suppose that* $\alpha \in \mathbb{Z}_+$ *and* $Q(x) := \int_0^x q(s)w(s)\,ds$. *Then*

$$(1)\ (V_{q,w}^{\alpha+1} f)(x) = q(x) \int_0^x \frac{(Q(x) - Q(t))^\alpha}{\alpha!} f(t)w(t)\,dt; \tag{2.1}$$

$$(2)\ ((\mathbb{I} - \lambda V_{q,w})^{-1} f)(x) = f(x) + \lambda q(x) \int_0^x e^{(Q(x)-Q(t))} f(t)w(t)\,dt;$$

$$(3)\ ((\mathbb{I} - \lambda V_{q,w}^\alpha)^{-1}) f(x) = f(x) + \lambda q(x) \int_0^x (Q(x) - Q(t))^\alpha E_{1/\alpha}(\lambda(Q(x) - Q(t))^\alpha; \alpha) f(t)w(t)\,dt, \tag{2.2}$$

where

$$E_{1/\alpha}(\lambda(x-t)^\alpha; \alpha) = \sum_{j=0}^{\infty} \frac{\lambda^j (x-t)^{\alpha j}}{\Gamma(\alpha + \alpha j)}$$

is the Mittag-Leffler function.

Proof. (1) We prove this statement by induction on α. Assume that relation (2.1) is true for α. Then, as a result of integration by parts, we get

$$\begin{aligned}
(V_{q,w}^{\alpha+1} f)(x) &= (V_{q,w}^\alpha V_{q,w} f)(x) \\
&= q(x) \int_0^x \frac{(Q(x) - Q(t))^{\alpha-1}}{(\alpha-1)!} (V_{q,w} f)(t)w(t)\,dt \\
&= q(x) \int_0^x \frac{(Q(x) - Q(t))^{\alpha-1}}{(\alpha-1)!} q(t)w(t) \left[\int_0^t f(s)w(s)\,ds \right] dt \\
&= q(x) \left[-\frac{(Q(x) - Q(t))^\alpha}{\alpha!} \int_0^t f(s)w(s)\,ds \,\Big|_0^x \right. \\
&\qquad \left. + \int_0^x \frac{(Q(x) - Q(t))^\alpha}{\alpha!} f(t)w(t)\,dt \right] \\
&= q(x) \int_0^x \frac{(Q(x) - Q(t))^\alpha}{\alpha!} f(t)w(t)\,dt.
\end{aligned}$$

Statements (2) and (3) follow from (1). □

Definition 2.1. *The operator* $T \in [X_1, X_2]$ *is called deformation if* $kerT = \{0\}$ *and* $\overline{TX_1} = X_2$.

Let $A \in [X_1]$ *and* $B \in [X_2]$. *The operators* A *and* B *are called quasisimilar if there exist deformations* $T_1 \in [X_1, X_2]$ *and* $T_2 \in [X_2, X_1]$ *such that* $T_1A = BT_1$ *and* $AT_2 = T_2B$.

The operators A *and* B *are called similar, if there exists an invertible operator* $T \in [X_1, X_2]$ *such that* $TA = BT$ *or* $A = T^{-1}BT$.

It turns out that, with the help of the property of quasisimilarity, the investigation of powers of the operator $V_{q,w}$ can be reduced to the case where $|q| \equiv 1$ and $w \equiv const$.

Theorem 2.1. *Let* $R(x) := \int_0^x |q(s)w(s)|ds$, $R(1) = 1$, $R^{-1}(x)$ *be the function inverse to* $R(x)$ ($R^{-1} \circ R = R \circ R^{-1} = x$) *and let* $r(x) :=$ sign$(qw)(R^{-1}(x))$. *Then*

(1) *the operators* $V_{q,w}$ *and* $V_{r,1}$ *are quasisimilar;*

(2) *the role of deformations* X *and* Y *intertwining the operators* $V_{q,w}$ *and* $V_{r,1}$, *can be played by the operators*

$$(Xf)(x) = q(x) \int\limits_0^{R(x)} f(t)\,dt, \tag{2.3}$$

$$(Yf)(x) = r(x) \int\limits_0^{R^{-1}(x)} w(t)f(t)\,dt, \tag{2.4}$$

i.e., $V_{q,w}X = XV_{r,1}$ *and* $YV_{q,w} = V_{r,1}Y$;

(3) the operators X *and* Y *satisfy the equalities*

$$XY = V_{q,w}^2, \qquad YX = V_{r,1}^2.$$

Proof. (2) We now show that the operators X and Y are bounded. Indeed,

$$\|(Xf)(x)\|_p^p = \int\limits_0^1 \left| q(x) \int\limits_0^{R(x)} f(t)dt \right|^p dx$$
$$\leqslant \int\limits_0^1 |q(x)|^p \left(\int_0^1 |f(t)|dt \right)^p dx \leqslant \|q(x)\|_p^p \|f\|_p^p,$$

$$\|(Yf)(x)\|_p^p = \int_0^1 \left| r(x) \int_0^{R^{-1}(x)} w(t)f(t)\,dt \right|^p dx$$

$$= \int_0^1 \left| \int_0^{R^{-1}(x)} w(t)f(t)\,dt \right|^p dx \leqslant \int_0^1 \left(\int_0^1 |w(t)f(t)|\,dt \right)^p dx \leqslant$$

$$\leqslant \|w(x)\|_{p'}^p \|f\|_p^p,$$

i.e.,

$$\|Xf\|_p \leqslant \|q\|_p \|f\|_p, \qquad \|Yf\|_p \leqslant \|w\|_{p'} \|f\|_p$$

and, hence, $\|X\| \leqslant \|q\|_p$ and $\|Y\| \leqslant \|w\|_{p'}$. The conditions $\ker X = \ker Y = \{0\}$ are checked trivially. We set

$$x_n(x) = x^n, \quad y_n(x) = q(x)\mathrm{sign}(qw)(x)\left(\int_0^x R(t)\,dt \right)^n \quad (n = 0, 1, \ldots).$$

Then

$$\mathrm{span}\{Xx_n : \ n \geqslant 1\} = \mathrm{span}\{q(x)R(x)^{n+1} : n \geqslant 1\} = L_p[0,1],$$

$$\mathrm{span}\{Yy_n : \ n \geqslant 1\} = \mathrm{span}\{r(x)\left(\int_0^{R^{-1}(x)} R(t)\,dt \right)^{n+1} : \ n \geqslant 1\} = L_p[0,1],$$

i.e., $\overline{\mathfrak{R}(X)} = \overline{\mathfrak{R}(Y)} = L_p[0,1]$. Thus, the operators X and Y are deformations. Further, we have

$$(V_{q,w}Xf)(x) = q(x) \int_0^x w(t)q(t) \int_0^{R(t)} f(s)\,ds\,dt$$

$$= q(x) \int_0^x \mathrm{sign}(qw)(t) \int_0^{R(t)} f(s)\,ds\,dR(t), \quad (2.5)$$

$$(YV_{q,w}f)(x) = r(x) \int_0^{R^{-1}(x)} w(t)q(t) \int_0^t w(s)f(s)\,ds\,dt$$

$$= r(x) \int_0^{R^{-1}(x)} \mathrm{sign}(qw)(t) \int_0^t w(s)f(s)\,ds\,dR(t). \quad (2.6)$$

In relations (2.5)–(2.6), we perform the change of variables $t = R^{-1}(t_1)$. This yields

$$(V_{q,w}Xf)(x) = q(x)\int_0^{R(x)} \operatorname{sign}(qw)(R^{-1}(t))\int_0^t f(s)\,ds\,dt = (XV_{r,1}f)(x),$$

$$(YV_{q,w}f)(x) = r(x)\int_0^x r(t)\int_0^{R^{-1}(t)} w(s)f(s)\,ds\,dt = (V_{r,1}Yf)(x).$$

Thus, the deformations X and Y intertwine the operators $V_{q,w}$ and $V_{r,1}$ (i.e., $V_{q,w}X = XV_{r,1}$ and $YV_{q,w} = V_{r,1}Y$) and, hence, the operators $V_{q,w}$ and $V_{r,1}$ are quasisimilar.

(3) We have

$$(XYf)(x) = q(x)\int_0^{R(x)} r(t)\int_0^{R^{-1}(t)} w(s)f(s)\,ds\,dt.$$

If we now perform the change of variables $t = R(t_1)$ and take into account the fact that $r(R(t))R'(t) = q(t)w(t)$, then we get

$$(XYf)(x) = q(x)\int_0^x q(t)w(t)\int_0^t f(s)w(s)\,ds\,dt = (V_{q,w}^2 f)(x).$$

Similarly, we find

$$(YXf)(x) = r(x)\int_0^{R^{-1}(x)} w(t)q(t)\int_0^{R(t)} f(s)\,ds\,dt$$
$$= r(x)\int_0^{R^{-1}(x)} r(R(t))\int_0^{R(t)} f(s)\,ds\,dR(t),$$

and the change of variables $t = R(t_1^{-1})$ gives $(YXf)(x) = (V_{r,1}^2 f)(x)$. □

Definition 2.2. ([17]) *A subspace E of a Banach space X is called a cyclic subspace of an operator $T \in [X]$ if* $\operatorname{span}\{T^nE : n \geq 0\} = X$*. A vector $f(\in X)$ is called cyclic if* $\operatorname{span}\{T^nf : n \geq 0\} = X$; $\operatorname{Cyc} T$ *is the set of all cyclic subspaces of the operator T.*

Definition 2.3. [16] *Denote*

$$\mu_T := \inf_E \{\dim E : E \textit{ is a cyclic subspace of the operator } T \textit{ in } X\}$$

and

$$\operatorname{disc} T := \sup_{E \in \operatorname{Cyc} T} \min\{\dim E' :\ E' \subset E,\ E' \in \operatorname{Cyc} T\}.$$

The number μ_T is called the spectral multiplicity of the operator T in X and $\operatorname{disc} T$ *is called the disc-characteristic of the operator T ("disc" stands for the Dimensionality of the Input Subspace of Control).*

Note that μ_T can be equal to ∞ and $\operatorname{disc} A \geq \mu_A$.

The operator T is called cyclic if $\mu_T = 1$.

It easy to see that if the operators A and B are quasisimilar, then $\mu_A = \mu_B$. We also note that both $\operatorname{disc} A$ and μ_A depend only on the lattice of invariant subspaces $\operatorname{Lat} A$ of the operator A.

Corollary 2.1. *Consider the operators X and Y defined by relations* (2.3) *and* (2.4). *In this case, the subspace $E \subset L_p[0,1] (F \subset L_p[0,1])$ is cyclic for the operator $V^\alpha_{q,w}(V^\alpha_{r,1})$ if and only if the subspace $\overline{YE}(\overline{XE})$ is cyclic for the operator $V^\alpha_{r,1}(V^\alpha_{q,w})$, i.e.,*

$$E \in \operatorname{Cyc} V^\alpha_{q,w} \Longleftrightarrow \overline{YE} \in \operatorname{Cyc} V^\alpha_{r,1}, \qquad F \in \operatorname{Cyc} V^\alpha_{r,1} \Longleftrightarrow \overline{XF} \in \operatorname{Cyc} V^\alpha_{q,w}.$$

Proof. We have

$$E \in \operatorname{Cyc} V_{q,w} \Rightarrow \overline{YE} \in \operatorname{Cyc} V_{r,1} \Rightarrow \overline{XYE} = \overline{V^2_{q,w} E} \in \operatorname{Cyc} V_{q,w} \Rightarrow E \in \operatorname{Cyc} V_{q,w},$$

$$F \in \operatorname{Cyc} V_{r,1} \Rightarrow \overline{XF} \in \operatorname{Cyc} V_{q,w} \Rightarrow \overline{YXF} = \overline{V^2_{r,1} F} \in \operatorname{Cyc} V_{r,1} \Rightarrow F \in \operatorname{Cyc} V_{r,1}.$$

If $\alpha \geqslant 2$, then we set

$$X_\alpha := V^{\alpha-2}_{q,w} X = V^{\alpha-2}_{r,1} X \quad \text{and} \quad Y_\alpha := V^{\alpha-2}_{r,1} Y = Y V^{\alpha-2}_{q,w}.$$

Thus, X_α and Y_α are deformations intertwining the operators $V_{q,w}$ and $V_{r,1}$ and we have $X_\alpha Y = V^\alpha_{q,w}$ and $Y_\alpha X = V^\alpha_{r,1}$. Hence, for $E \in \operatorname{Cyc} V^\alpha_{q,w}$ and $F \in \operatorname{Cyc} V^\alpha_{r,1}$, we can write

$$\overline{YE} \in \operatorname{Cyc} V^\alpha_{r,1}, \quad \overline{Y_\alpha E} \in \operatorname{Cyc} V^\alpha_{r,1}, \quad \overline{XF} \in \operatorname{Cyc} V^\alpha_{q,w}, \quad \overline{X_\alpha F} \in \operatorname{Cyc} V^\alpha_{q,w}$$

and, therefore, the following implications are true:

$$E \in \mathrm{Cyc} V^{\alpha}_{q,w} \Rightarrow \overline{YE} \in \mathrm{Cyc} V^{\alpha}_{r,1} \Rightarrow \overline{X_\alpha YE} \in \mathrm{Cyc} V^{\alpha}_{q,w} \\ \Rightarrow \overline{V^{\alpha}_{q,w} E} \in \mathrm{Cyc} V^{\alpha}_{q,w} \Rightarrow E \in \mathrm{Cyc} V^{\alpha}_{q,w},$$

$$F \in \mathrm{Cyc} V^{\alpha}_{r,1} \Rightarrow \overline{XF} \in \mathrm{Cyc} V^{\alpha}_{q,w} \Rightarrow \overline{Y_\alpha XF} \in \mathrm{Cyc} V^{\alpha}_{r,1} \\ \Rightarrow \overline{V^{\alpha}_{r,1} F} \in \mathrm{Cyc} V^{\alpha}_{r,1} \Rightarrow F \in \mathrm{Cyc} V^{\alpha}_{r,1}.$$

□

Corollary 2.2. *Let $q, w \in C[0,1]$, let $qw > 0$, and let $\int_0^1 q(t)w(t)dt = 1$. Then the operators $V^{\alpha}_{q,w}$ and J^α are similar and, hence, the operator $V^{\alpha}_{q,w}$ is unicellular.*

Proof. We now show that the operator

$$(Tf)(x) = q(x) f\left(\int_0^x q(t)w(t)\, dt\right) \tag{2.7}$$

is bounded together with the inverse operator and $TJ^\alpha = V^{\alpha}_{q,w} T$. Indeed, since $q, w \in C[0,1]$ and $qw > 0$, we can write

$$0 < m := \min_{x\in[0,1]} \frac{|q(x)|^p}{q(x)w(x)} < M := \max_{x\in[0,1]} \frac{|q(x)|^p}{q(x)w(x)} < \infty$$

and

$$\|Tf\|_p^p = \int_0^1 \left| f\left(\int_0^x q(t)w(t)\, dt\right)\right|^p \frac{|q(x)|^p}{q(x)w(x)}\, d\int_0^x q(t)w(t)\, dt.$$

Therefore,

$$m\|f\|_p^p = m\int_0^1 \left| f\left(\int_0^x q(t)w(t)\, dt\right)\right|^p d\int_0^x q(t)w(t)\, dt < \|Tf\|_p^p \\ < M\int_0^1 \left| f\left(\int_0^x q(t)w(t)\, dt\right)\right|^p d\int_0^x q(t)w(t)\, dt = M\|f\|_p^p.$$

This means that the operators T and T^{-1} are bounded. □

Remark 2.1. Note that an operator T of the form (2.7) is the Liouville transformation which, in some cases, enables one to reduce the Sturm–Liouville equation with nonsummable potential to an equation with summable potential [2].

Remark 2.2. In the case where $q, w \in AC[0,1]$, the similarity of the operators $V_{q,w}$ and cJ follows from the general Malamud's result (see [14]) on the similarity of the operator K ($(Kf)(x) = \int_0^x k(x,t)f(t)dt$) to the operator of integration J.

Note that, in his recent work [4], Gubreev has studied the problem of similarity of dissipative operators K with singular kernels to the operator of integration.

Remark 2.3. The Liouville transformation can also be useful in some other cases. Thus, let $w(x) \in L_2(\mathbb{R}_+)$ be a step function with finitely many jumps. In [9], it is shown that the operator $w(x)i\frac{d}{dx}$ acting in $L_2(\mathbb{R}_+)$ is similar to a self-adjoint operator if and only if the function $w(x)$ does not change the sign. In the case where $w(x) > 0$, the role of this operator is played by the operator $w(x)^{\frac{1}{2}}i\frac{d}{dx}w(x)^{\frac{1}{2}}$ acting in the same space. It is easy to see that if $w(x)$ is a positive bounded function separated from zero, then the operator $T : f \to f(\int_0^x \frac{1}{w(s)}ds)$ realizes the similarity of the operators $w(x)i\frac{d}{dx}$ and $i\frac{d}{dx}$. Indeed, since

$$\|Tf\|_2^2 = \int_0^\infty \left| f\left(\int_0^x \frac{1}{w(t)}\,dt \right) \right|^2 w(x) d\left(\int_0^x \frac{1}{w(t)}\,dt \right),$$

we have

$$\inf_{x\in\mathbb{R}_+} w(x)\|f\|_2^2 \le \|Tf\|_2^2 \le \sup_{x\in\mathbb{R}_+} w(x)\|f\|_2^2$$

and, therefore, the operators T and T^{-1} are bounded. Further, we get

$$\Big(wi\frac{d}{dx}Tf\Big)(x) = \Big(Ti\frac{d}{dx}f\Big)(x) = if'\left(\int_0^x \frac{1}{w(t)}\,dt \right)$$

and, hence, all operators of the form $w(x)i\frac{d}{dx}$, where $w(x)$ is a positive bounded function separated from zero, are similar to the operator $i\frac{d}{dx}$.

By virtue of Corollary 2.1, in order to describe the set $\mathrm{Cyc}V_{q,w}^\alpha$, it suffices to consider the case when $w \equiv 1$ and the function q takes the values ± 1. For each finite partition

$$\pi = \{0 = a_0 < a_1 \cdots < a_{n-1} < a_n = 1\} \tag{2.8}$$

of the segment $[0,1]$, we define a function $d_\pi(x)$ as follows:

$$d_\pi(x) := \begin{cases} +1, & x \in [a_{2i}, a_{2i+1}], \quad i = 0,1,\dots,[\frac{n-1}{2}], \\ -1, & x \in [a_{2i-1}, a_{2i}], \quad i = 0,1,\dots,[\frac{n}{2}]. \end{cases} \tag{2.9}$$

Then

(I) either there exists a finite partition π of the form (2.8) such that the function $q(x)$ is equivalent to a function $d_\pi(x)$ (or $-d_\pi(x)$) of the form (2.9) or

(II) the function $q(x)$ is not equivalent to a function $d_\pi(x)$ (or $-d_\pi(x)$) of the form (2.9) for any finite partition π of the form (2.8).

We consider these two cases separately.

3. Cyclic Subspaces. Case I

In this section, we describe cyclic subspaces of the operator $V^\alpha_{q,1}$ in the case where there exists a finite partition π of the form (2.8) such that the function $q(x)$ is equivalent to a function $d_\pi(x)$ of the form (2.9). For this purpose, need the following two lemmas:

Lemma 3.1. *Let $0 = a_0 < a_1 \cdots < a_{n-1} < a_n = 1$. For $1 \leqslant k \leqslant m \leqslant n$, we define the numbers $A_{k,m}$ as follows:*

$$A_{k,m} := \begin{cases} \sum\limits_{l=i+1}^{j} a_{2l-1} - \sum\limits_{l=i+1}^{j} a_{2l} & \text{for } k = 2i+1, \quad m = 2j+1, \\ \sum\limits_{l=i+1}^{j} a_{2l-1} - \sum\limits_{l=i}^{j} a_{2l} & \text{for } k = 2i, \quad m = 2j+1, \\ \sum\limits_{l=i+1}^{j} a_{2l-1} - \sum\limits_{l=i+1}^{j-1} a_{2l} & \text{for } k = 2i+1, \quad m = 2j, \\ \sum\limits_{l=i+1}^{j} a_{2l-1} - \sum\limits_{l=i}^{j-1} a_{2l} & \text{for } k = 2i, \quad m = 2j \end{cases}$$

$$= \sum_{l=[\frac{k}{2}]+1}^{[\frac{m}{2}]} a_{2l-1} - \sum_{l=[\frac{k+1}{2}]}^{[\frac{m-1}{2}]} a_{2l}.$$

Then

(1) $0 < (-1)^m A_{k,m} < 1$*;*

(2) $0 < (-1)^{m-1} A_{k,m-1} = a_{m-1} + (-1)^{m+1} A_{k,m} < a_m + (-1)^{m+1} A_{k,m} < a_m < 1$*;*

(3) $a_k + (-1)^k A_{k,m} = (-1)^k A_{k+1,m}$, $\quad a_{k-1} + (-1)^k A_{k,m} = (-1)^k A_{k-1,m}$*;*

(4) $(-1)^{k+m} a_{k-1} + (-1)^m A_{k,m} \in (0,1)$, $(-1)^{k+m} a_k + (-1)^m A_{k,m} \in (0,1)$.

Proof. (1) We have

$$(-1)^m A_{k,m} = \begin{cases} -\sum\limits_{[\frac{k}{2}]+1}^{j} a_{2l-1} + \sum\limits_{[\frac{k+1}{2}]}^{j} a_{2l}, & m = 2j+1, \\ -\sum\limits_{[\frac{k}{2}]+1}^{j} a_{2l-1} + \sum\limits_{[\frac{k+1}{2}]}^{j-1} a_{2l}, & m = 2j. \end{cases}$$

Since $[\frac{k+1}{2}] \le [\frac{k}{2}] + 1$, we get

$$0 < \begin{cases} \sum\limits_{[\frac{k}{2}]+1}^{j} (a_{2l} - a_{2l-1}), & m = 2j+1, \\ \sum\limits_{[\frac{k+1}{2}]}^{j-1} (a_{2l+1} - a_{2l}), & m = 2j \end{cases} \leqslant (-1)^m A_{k,m}$$

$$\le \begin{cases} \sum\limits_{[\frac{k+1}{2}]}^{j} (a_{2l} - a_{2l-1}), & m = 2j+1, \\ \sum\limits_{[\frac{k}{2}]}^{j-1} (a_{2l+1} - a_{2l}), & m = 2j \end{cases} < 1.$$

The equalities in (2) and (3) are proved by the direct substitution and both inequalities in (2) follow from (1).

(4) We have

$$\begin{aligned} (-1)^{k+m} a_{k-1} + (-1)^m A_{k,m} &= (-1)^{m+k}(a_{k-1} + (-1)^k A_{k,m}) \\ &= (-1)^{m+k}((-1)^k A_{k-1,m}) = (-1)^m A_{k-1,m}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} (-1)^{k+m} a_k + (-1)^m A_{k,m} &= (-1)^{m+k}(a_k + (-1)^k A_{k,m}) \\ &= (-1)^{m+k}((-1)^k A_{k+1,m}) = (-1)^m A_{k+1,m}. \end{aligned} \tag{3.2}$$

As follows from assertion (1), the values of expressions (3.1)–(3.2) belong to the segment $[0, 1]$. □

In what follows, for the sake of brevity, we set ${E_\alpha}^{\lambda,t} := E_{1/\alpha}(\lambda t^\alpha; \alpha)$. Note that

$${E_\alpha}^{\lambda,t} := E_{1/\alpha}(\lambda t^\alpha; \alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha + \alpha k)} = \frac{1}{\alpha t^{\alpha-1}} \sum_{k=1}^{\alpha} \varepsilon_k e^{\varepsilon_k \lambda^{\frac{1}{\alpha}} t}, \tag{3.3}$$

where $\varepsilon_k = e^{2\pi i k/\alpha}$ are different roots of 1. Thus, in particular, if $\alpha = 1$, then ${E_\alpha}^{\lambda,t} = e^{\lambda t}$.

Lemma 3.2. *(1) Let* $f_1, f_2 \in L_p[0,1]$ *and let*

$$\int_0^1 {E_\alpha}^{\lambda,t} f_1(t)\,dt + \int_0^1 {E_\alpha}^{\lambda,-t} f_2(t)\,dt = 0, \qquad \lambda \in \mathbb{C}. \tag{3.4}$$

Then

(a) $f_1(t) + f_2(t) = 0$ *for almost all* $t \in [0,1]$ *if* α *is even;*

(b) $f_1(t) = f_2(t) = 0$ *for almost all* $t \in [0,1]$ *if* α *is odd.*

(2) Let $f \in L_p[0,1]$ *and let* $g \in L_{p'}[0,1]$. *Then*

$$\begin{aligned}\int_0^1 \int_0^1 f(t){E_\alpha}^{\lambda,x-t}\,dt\,g(x)\,dx &= \int_0^1 {E_\alpha}^{\lambda,t} \int_t^1 f(x-t)g(x)\,dx\,dt \\ &\quad + \int_0^1 {E_\alpha}^{\lambda,-t} \int_t^1 f(1-x+t)g(1-x)\,dx\,dt. \end{aligned} \tag{3.5}$$

Proof. We rewrite equality (3.4) in the form

$$\int_0^1 {E_\alpha}^{\lambda,t} f_1(t)\,dt + \int_{-1}^0 {E_\alpha}^{\lambda,-t} f_2(-t)\,dt = \int_{-1}^1 {E_\alpha}^{\lambda,t}(\tilde{f}_1(t) + \tilde{f}_2(t))\,dt = 0,$$

where

$$\tilde{f}_1(t) = \begin{cases} 0, & t \in [-1,0], \\ f_1(t), & t \in [0,1] \end{cases}, \qquad \tilde{f}_2(t) = \begin{cases} f_2(-t), & t \in [0,-1], \\ 0, & t \in [0,1]. \end{cases}$$

If α is even, then ${E_\alpha}^{\lambda,t} = {E_\alpha}^{\lambda,-t}$ and the subspace

$$E := \operatorname{span}\{{E_\alpha}^{\lambda,t} : \lambda \in \mathbb{C}, t \in [-1,1]\} \subset L_p[-1,1]$$

coincides with the subspace of functions even on the segment $[-1,1]$. Therefore, the function $\tilde{f}_1(t) - \tilde{f}_2(t)$ must be odd on $[-1,1]$ and, hence, $f_1(t) + f_2(t) = 0$ for almost all $t \in [0,1]$.

If α is odd, then $E = L_p[-1,1]$ and, therefore, $\tilde{f}_1(t) - \tilde{f}_2(t) = 0$ for all $t \in [-1,1]$. Since $\operatorname{supp}\tilde{f}_1 \cap \operatorname{supp}\tilde{f}_2 = 0$, we have $f_1(t) = f_2(t) = 0$ for almost all $t \in [0,1]$.

(2) We have

$$\int_0^1\int_0^1 f(t){E_\alpha}^{\lambda,x-t}\,dt\,g(x)\,dx=\int_0^1\int_0^x+\int_0^1\int_x^1=:I_1+I_2.$$

By changing the order of integration in I_1 and performing the change of variables $0\leqslant x=1-x_1\leqslant 1$ and $x\leqslant t=1-t_1\leqslant 1$ in I_2, we arrive at the required assertion. □

Remark 3.1. Note that, for even α, relation (3.5) takes the form

$$\int_0^1\int_0^1 f(t){E_\alpha}^{\lambda,x-t}\,dt\,g(x)\,dx$$
$$=\int_0^1 {E_\alpha}^{\lambda,t}\left(\int_t^1 f(x-t)g(x)\,dx+\int_t^1 f(1-x+t)g(1-x)\,dx\right)dt.$$

For each function $f\in L_p[0,1]$, we now define the following functions:

$$\tilde{f}^j(x):=\chi_{[a_{j-1},a_j]}(x)f(a_j-x)+\chi_{[a_j,a_{j+1}]}(x)f(a_j+x)\quad (j=1,\dots,n-1)$$

and $\tilde{f}^0(x):=\chi_{[0,a_1]}(x)f(x)$.

For $f\in L_p[0,1]$, we denote

$$E_{V_{q,1}^\alpha}(f):=\operatorname{span}\{V_{q,1}^{\alpha n}f:n\geqslant 0\}.$$

Further, by $E_{V_{q,1}^\alpha}(f)^\perp$ we denote the annihilator of the subspace $E_{V_{q,1}^\alpha}(f)$, i.e., the space of functions $g\in L_{p'}[0,1]$ such that

$$\int_0^1 (V_{q,1}^{\alpha n}f)(x)g(x)\,dx=0\qquad (n=0,1,\dots).\tag{3.6}$$

Note that, for $p=2$, the subspace $E_{V_{q,1}^\alpha}(f)^\perp$ is the orthogonal complement to the subspace $E_{V_{q,1}^\alpha}(f)$.

Proposition 3.1. *Assume that $q(x)$ is equivalent to a function $d_\pi(x)$ of the form* (2.9)*. Then $g\in E_{V_{q,1}^\alpha}(f)^\perp\subset L_{p'}[0,1]$ if and only if*

(1)

$$\sum_{1\leqslant i\leqslant[\frac{n+1}{2}]}\tilde{f}^{2i-2}(x)*G_{2i-1}(x)=0\tag{3.7}$$

and

$$\sum_{1\leqslant i\leqslant[\frac{n+1}{2}]} \tilde{f}^{2i-1}(x) * G_{2i}(x) = 0 \tag{3.8}$$

for odd α *and*

(2)

$$\sum_{1\leqslant i\leqslant[\frac{n+1}{2}]} \tilde{f}^{2i-2}(x) * G_{2i-1}(x) + \sum_{1\leqslant i\leqslant[\frac{n+1}{2}]} \tilde{f}^{2i-1}(x) * G_{2i}(x) = 0,$$

for even α, *where*

$$G_{2i-1}(x) = \sum_{m\geqslant 2i-1} g_m((-1)^{m-1}(1-x-2A_{2i-1,m}+a_{2i-2})), \quad (i \geqslant 1), \tag{3.9}$$

$$G_{2i}(x) = \sum_{m\geqslant 2i} g_m((-1)^{m}(1-x+2A_{2i,m}+a_{2i-1})), \quad (i \geqslant 1). \tag{3.10}$$

Proof. For $1 \leqslant k$ and $m \leqslant n$, we set

$$f_k(x) = \chi_{[a_{k-1},a_k]}(x)f(x) \quad \text{and} \quad g_m(x) = \chi_{[a_{m-1},a_m]}(x)g(x), \quad x \in [0,1]. \tag{3.11}$$

We also set $f_0 \equiv 0$ and extend the functions f_k and g_m by zero outside the segment $[0,1]$. Then $f = f_1 + \cdots + f_n$ and $g = g_1 + \cdots + g_n$ and, in view of relation (2.2) and notation (3.3), condition (3.6) can be rewritten in the form

$$\int_0^1 q(x) \int_0^x f(t) E_\alpha^{\lambda, \int_t^x q(s)\,ds} dt\, g(x)\, dx$$

$$= \sum_{k,m=1}^{n} \int_0^1 (-1)^{m+1} \int_0^x f_k(t) E_\alpha^{\lambda, \int_t^x q(s)\,ds} dt\, g_m(x)\, dx$$

$$=: \sum_{1\leqslant k\leqslant m\leqslant n} I_{k,m} + \sum_{1\leqslant m<k\leqslant n} I_{k,m} =: S_1 + S_2 = 0.$$

Since $\operatorname{supp} f_k \subset [a_{k-1}, a_k]$ and $\operatorname{supp} g_m \subset [a_{m-1}, a_m]$, for $k > m$, we obtain

$$\operatorname{supp}\left(g_m(x) \int_0^x f_k(t) E_\alpha^{\lambda, \int_t^x q(s)\,ds} dt\right) \subset [a_{k-1}, 1] \cap [a_{m-1}, a_m] = \{\emptyset\}.$$

Therefore, $S_2 = 0$. It is easy to see that

$$\int_t^x q(s)\,ds = \begin{cases} 2\sum_{l=i+1}^{j} a_{2l-1} - 2\sum_{l=i+1}^{j} a_{2l} + x - t, & t \in [a_{2i}, a_{2i+1}], x \in [a_{2j}, a_{2j+1}], \\ 2\sum_{l=i+1}^{j} a_{2l-1} - 2\sum_{l=i}^{j} a_{2l} + x + t, & t \in [a_{2i-1}, a_{2i}], x \in [a_{2j}, a_{2j+1}], \\ 2\sum_{l=i+1}^{j} a_{2l-1} - 2\sum_{l=i+1}^{j-1} a_{2l} - x - t, & t \in [a_{2i}, a_{2i+1}], x \in [a_{2j-1}, a_{2j}], \\ 2\sum_{l=i+1}^{j} a_{2l-1} - 2\sum_{l=i}^{j-1} a_{2l} - x + t, & t \in [a_{2i-1}, a_{2i}], x \in [a_{2j-1}, a_{2j}]. \end{cases}$$

Hence, for $t \in [a_{k-1}, a_k]$ and $x \in [a_{m-1}, a_m]$, we can write

$$\begin{aligned} \int_t^x q(s)\,ds &= 2\sum_{l=[\frac{k}{2}]+1}^{[\frac{m}{2}]} a_{2l-1} - 2\sum_{l=[\frac{k+1}{2}]}^{[\frac{m-1}{2}]} a_{2l} + (-1)^k t + (-1)^{m+1} x \\ &= 2A_{k,m} + (-1)^k t + (-1)^{m+1} x. \end{aligned}$$

We now consider the terms in S_1:

(i) For $k = m$, we have

$$\begin{aligned} I_{m,m} &= \int_0^1 (-1)^{m+1} \int_0^x f_m(t) E_\alpha{}^{\lambda,(-1)^{m+1}(x-t)}\,dt\, g_m(x)\,dx \\ &= \int_0^1 (-1)^{m+1} E_\alpha{}^{\lambda,(-1)^{m+1}t} F_{m,m}(t)\,dt, \end{aligned}$$

where

$$F_{m,m} = \int_t^1 f_m(x-t) g_m(x)\,dx. \tag{3.12}$$

(ii) The case where $I_{k,m}$ for $m > k$ is studied in more details; thus, we have

$$\begin{aligned} I_{k,m} &= \int_0^1 (-1)^{m+1} \int_0^x f_k(t) E_\alpha{}^{\lambda,(2A_{k,m}+(-1)^k t+(-1)^{m+1}x)}\,dt\, g_m(x)\,dx \\ &= \int_{a_{m-1}}^{a_m} (-1)^{m+1} \int_{a_{k-1}}^{a_k} f_k(t) E_\alpha{}^{\lambda,(2A_{k,m}+(-1)^k t+(-1)^{m+1}x)}\,dt\, g_m(x)\,dx. \end{aligned}$$

Further, we perform the change of variables

$$a_{m-1} \leqslant x = x_1 + (-1)^m A_{k,m} \leqslant a_m,$$

$$a_{k-1} \leqslant t = (-1)^{k+m} t_1 + (-1)^{k+1} A_{k,m} \leqslant a_k$$

and obtain

$$I_{k,m} = \int\limits_{a_{m-1}+(-1)^{m+1}A_{k,m}}^{a_m+(-1)^{m+1}A_{k,m}} (-1)^{m+1}(-1)^{k+m} \int\limits_{(-1)^{k+m}a_{k-1}+(-1)^m A_{k,m}}^{(-1)^{k+m}a_k+(-1)^m A_{k,m}}$$

$$f_k((-1)^{k+m}t + (-1)^{k+1}A_{k,m}) E_\alpha{}^{\lambda,(-1)^{m+1}(x-t)} dt g_m(x + (-1)^m A_{k,m}) dx.$$

Since, in view of assertions (2) and (4) in Lemma 3.1, the limits of integration belong to the segment $[0, 1]$ and, by virtue of relation (3.11), the integrand is equal to zero outside the limits of integration, we get

$$I_{k,m} = \int\limits_0^1 \int\limits_0^1 f_k((-1)^{k+m}t + (-1)^{k+1}A_{k,m}) E_\alpha{}^{\lambda,(-1)^{m+1}(x-t)}\, dt \times g_m(x + (-1)^m A_{k,m})\, dx.$$

By virtue of assertion (2) in Lemma 3.2, we find

$$I_{k,m} = \int\limits_0^1 E_\alpha{}^{\lambda,(-1)^{m+1}t} F_{k,m}(t)\, dt + \int\limits_0^1 E_\alpha{}^{\lambda,(-1)^m t} G_{k,m}(t)\, dt, \qquad (3.13)$$

where

$$F_{k,m}(t) := \int\limits_t^1 f_k((-1)^{k+m}(x-t) + (-1)^{k+1}A_{k,m}) g_m(x + (-1)^m A_{k,m})\, dx, \qquad (3.14)$$

$$G_{k,m}(t) := \int\limits_t^1 f_k((-1)^{k+m}(1-x+t) + (-1)^{k+1}A_{k,m}) \times g_m(1 - x + (-1)^m A_{k,m})\, dx.$$

To transform the integrals $F_{k,m}(t)$ and $G_{k,m}(t)$, we consider the cases where $k+m$ is even and odd separately:

(ii1) $k+m$ is even.

First, we study the integral $F_{k,m}(t)$. In this integral, we perform the change of variables

$$t < x = x_1 + (-1)^k A_{k,m} + a_{k-1} = x_1 + (-1)^m A_{k,m} + a_{k-1} < 1.$$

This gives

$$F_{k,m}(t) = \int\limits_{t-(-1)^k A_{k,m}-a_{k-1}}^{1-(-1)^k A_{k,m}-a_{k-1}} f_k(x-t+a_{k-1}) g_m(x+2(-1)^m A_{k,m}+a_{k-1}) dx.$$

According to Lemma 3.1, we have

$$-(-1)^k A_{k,m} - a_{k-1} = -(-1)^{k+m}(a_{k-1} + (-1)^m A_{k,m}) < 0.$$

Therefore, $g_m(x+2(-1)^m A_{k,m}+a_{k-1}) = 0$ for $1 > x > 1-(-1)^k A_{k,m} - a_{k-1}$ since $x+2(-1)^m A_{k,m}+a_{k-1} > 1+(-1)^m A_{k,m} > 1$. Similarly, $f_k(x-t+a_{k-1}) = 0$ for $x < t$ because $x - t + a_{k-1} < a_{k-1}$ and, consequently,

$$F_{k,m}(t) = \int\limits_t^1 f_k(x - t + a_{k-1}) g_m(x + 2(-1)^m A_{k,m} + a_{k-1}) dx. \qquad (3.15)$$

We now consider the integral $G_{k,m}(t)$. In this integral, we perform the change of variables

$$t < x = x_1 + 1 - (-1)^k A_{k,m} - a_k = x_1 + 1 - (-1)^m A_{k,m} - a_k < 1.$$

This gives

$$G_{k,m}(t) = \int\limits_{t-1+(-1)^k A_{k,m}+a_k}^{(-1)^k A_{k,m}+a_k} f_k(t - x + a_k) g_m(-x + 2(-1)^m A_{k,m} + a_k)\, dx.$$

By virtue of Lemma 3.1, we have

$$(-1)^k A_{k,m} + a_k = (-1)^{k+m} a_k + (-1)^m A_{k,m} \in [0,1]$$

and, therefore, $g_m(-x+2(-1)^m A_{k,m}+a_k) = 0$ for $1 > x > (-1)^k A_{k,m} + a_k$ in view of the fact that $-x + 2(-1)^m A_{k,m} + a_k < (-1)^m A_{k,m} = a_{m-1} + (-1)^m A_{k,m-1} < a_{m-1}$. Similarly, $f_k(t - x + a_k) = 0$ for $x < t$ because $t - x + a_k > a_k$ and, hence,

$$G_{k,m}(t) = \int\limits_t^1 f_k(t - x + a_k) g_m(-x + 2(-1)^m A_{k,m} + a_k) dx \qquad (3.16)$$

(ii2) $k+m$ is odd.

First, we consider the integral $F_{k,m}(t)$. In this integral, we perform the change of variables

$$t < x = x_1 - (-1)^k A_{k,m} - a_k = x_1 + (-1)^m A_{k,m} - a_k < 1$$

and, as a result, obtain

$$F_{k,m}(t) = \int\limits_{t+(-1)^k A_{k,m}+a_k}^{1+(-1)^k A_{k,m}+a_k} f_k(t-x+a_k) g_m(x+2(-1)^m A_{k,m} - a_k)\, dx.$$

According to Lemma 3.1, we have

$$(-1)^k A_{k,m} + a_k = (-1)^{k+m}((-1)^{k+m} a_k + (-1)^m A_{k,m}) < 0.$$

Therefore, $g_m(x+2(-1)^m A_{k,m} - a_k) = 0$ for $1 > x > 1 + (-1)^k A_{k,m} + a_k$ since $x + 2(-1)^m A_{k,m} - a_k > 1 + (-1)^m A_{k,m} > 1$. Similarly, $f_k(t-x+a_k) = 0$ for $x < t$ because $x - t + a_k > a_k$ and, consequently,

$$F_{k,m}(t) = \int\limits_t^1 f_k(t-x+a_k) g_m(x+2(-1)^m A_{k,m} - a_k)\, dx. \qquad (3.17)$$

We now consider the integral $G_{k,m}(t)$. In this integral, we perform the change of variables

$$t < x = x_1 + 1 + (-1)^k A_{k,m} + a_{k-1} = x_1 + 1 - (-1)^m A_{k,m} + a_{k-1} < 1.$$

This gives

$$G_{k,m}(t) = \int\limits_{t-1-(-1)^k A_{k,m}-a_{k-1}}^{-(-1)^k A_{k,m}-a_{k-1}} f_k(x-t+a_{k-1}) g_m(-x+2(-1)^m A_{k,m} - a_{k-1})\, dx.$$

By virtue of Lemma 3.1, we have

$$-(-1)^k A_{k,m} - a_{k-1} = (-1)^{k+m} a_{k-1} + (-1)^m A_{k,m} \in [0,1].$$

Hence, $g_m(-x + 2(-1)^m A_{k,m} - a_{k-1}) = 0$ for $1 > x > -(-1)^k A_{k,m} - a_{k-1}$ because $-x + 2(-1)^m A_{k,m} - a_{k-1} < (-1)^m A_{k,m} = a_{m-1} + (-1)^m A_{k,m-1} < a_{m-1}$ and, similarly, $f_k(x-t+a_{k-1}) = 0$ for $x < t$ because $x - t + a_{k-1} < a_{k-1}$. Therefore,

$$G_{k,m}(t) = \int\limits_t^1 f_k(x-t+a_{k-1}) g_m(-x+2(-1)^m A_{k,m} - a_{k-1})\, dx. \quad (3.18)$$

In relations (3.15)–(3.18), we perform the change of variables $t < x = 1 - x_1 < 1$. Then, for even $k+m$, we obtain

$$F_{k,m}(1-t) = \int_0^t f_k(t - x + a_{k-1}) g_m(1 - x + 2(-1)^m A_{k,m} + a_{k-1})\, dx,$$

$$G_{k,m}(1-t) = \int_0^t f_k(x - t + a_k) g_m(-1 + x + 2(-1)^m A_{k,m} + a_k)\, dx.$$

Further, if $k+m$ is odd, then we get

$$F_{k,m}(1-t) = \int_0^t f_k(x - t + a_k) g_m(1 - x + 2(-1)^m A_{k,m} - a_k)\, dx,$$

$$G_{k,m}(1-t) = \int_0^t f_k(t - x + a_{k-1}) g_m(-1 + x + 2(-1)^m A_{k,m} - a_{k-1})\, dx.$$

Note that, for $k = m$, the definition of the function $F_{k,m}$ (3.14) coincides with relation (3.12). Therefore, in what follows, we assume that the functions $F_{k,m}$ are given by relation (3.14) for $k \leqslant m$. Thus, for $i \leqslant j$, we can write

$$F_{2i-1,2j-1}(1-t) = f_{2i-1}(a_{2i-2} + x) * g_{2j-1}(1 - x - 2A_{2i-1,2j-1} + a_{2i-2}),$$

$$F_{2i,2j}(1-t) = f_{2i}(a_{2i-1} + x) * g_{2j}(1 - x + 2A_{2i,2j} + a_{2i-1}),$$

$$F_{2i-1,2j}(1-t) = f_{2i-1}(a_{2i-1} - x) * g_{2j}(1 - x + 2A_{2i-1,2j} - a_{2i-1}),$$

$$F_{2i-2,2j-1}(1-t) = f_{2i-2}(a_{2i-2} - x) * g_{2j-1}(1 - x - 2A_{2i-1,2j-1} + a_{2i-2}),$$

$$G_{2i-1,2j+1}(1-t) = f_{2i-1}(a_{2i-1} - x) * g_{2j+1}(-1 + x - 2A_{2i-1,2j+1} + a_{2i-1}),$$

$$G_{2i-2,2j}(1-t) = f_{2i-2}(a_{2i-2} - x) * g_{2j}(-1 + x + 2A_{2i-1,2j} - a_{2i-2}),$$

$$G_{2i,2j+1}(1-t) = f_{2i}(a_{2i-1} + x) * g_{2j+1}(-1 + x - 2A_{2i,2j+1} - a_{2i-1}),$$

$$G_{2i-1,2j}(1-t) = f_{2i-1}(a_{2i-2} + x) * g_{2j}(-1 + x + 2A_{2i-1,2j} - a_{2i-2}).$$

In view of relation (3.13), the equation $S_1 = \sum\limits_{1\leqslant k\leqslant m\leqslant n} I_{k,m} = 0$ can be rewritten in the form

$$\begin{aligned}
S_1 = \sum_{1\leqslant m\leqslant n} I_{m,m} + \sum_{1\leqslant k<m\leqslant n} I_{k,m} \\
&= \sum_{1\leqslant k\leqslant m\leqslant n} \int\limits_0^1 E_\alpha{}^{\lambda,(-1)^{m+1}t} F_{k,m}(t)\,dt \\
&+ \sum_{1\leqslant k<m\leqslant n} \int\limits_0^1 E_\alpha{}^{\lambda,(-1)^{m}t} G_{k,m}(t)\,dt \\
= \int\limits_0^1 E_\alpha{}^{\lambda,t}\Bigg(\sum_{1\leqslant k\leqslant 2j-1\leqslant n} F_{k,2j-1}(t) + \sum_{1\leqslant k<2j\leqslant n} G_{k,2j}(t)\Bigg)dt \\
+ \int\limits_0^1 E_\alpha{}^{-\lambda,t}\Bigg(\sum_{1\leqslant k\leqslant 2j\leqslant n} F_{k,2j}(t) + \sum_{1\leqslant k<2j-1\leqslant n} G_{k,2j-1}(t)\Bigg)dt \\
&=: \int\limits_0^1 E_\alpha{}^{\lambda,t} H_+(t)\,dt + \int\limits_0^1 E_\alpha{}^{\lambda,-t} H_-(t)\,dt = 0. \qquad (3.19)
\end{aligned}$$

We now present the detailed expressions for the functions $H_+(t)$ and $H_-(t)$ obtained by using the fact that $F_{0,2j-1} \equiv G_{0,2j} \equiv 0$ (since $f_0 \equiv 0$):

$$\begin{aligned}
H_+(t) &:= \sum_{1\leqslant k\leqslant 2j-1\leqslant n} F_{k,2j-1}(t) + \sum_{1\leqslant k<2j\leqslant n} G_{k,2j}(t) \\
&= \sum_{1\leqslant i\leqslant j\leqslant n/2} \big(F_{2i-2,2j-1}(t) + F_{2i-1,2j-1}(t) + G_{2i-2,2j}(t) + G_{2i-1,2j}(t)\big) \\
&= \sum_{1\leqslant i\leqslant j\leqslant n/2} \Big(f_{2i-2}(a_{2i-2} - x) * g_{2j-1}(1 - x - 2A_{2i-1,2j-1} + a_{2i-2}) \\
&\quad + f_{2i-1}(a_{2i-2} + x) * g_{2j-1}(1 - x - 2A_{2i-1,2j-1} + a_{2i-2}) \\
&\quad + f_{2i-2}(a_{2i-2} - x) * g_{2j}(-1 + x + 2A_{2i-1,2j} - a_{2i-2}) \\
&\quad + f_{2i-1}(a_{2i-2} + x) * g_{2j}(-1 + x + 2A_{2i-1,2j} - a_{2i-2})\Big) \\
&= \sum_{1\leqslant i\leqslant n/2} \Big[(f_{2i-2}(a_{2i-2} - x) + f_{2i-1}(a_{2i-2} + x))
\end{aligned}$$

$$* \sum_{m\geqslant 2i-1} g_m((-1)^{m-1}(1-x-2A_{2i-1,m}+a_{2i-2}))\Big]$$

$$= \sum_{1\leqslant i\leqslant n/2} \tilde{f}^{2i-2}(x) * G_{2i-1}(x)$$

and

$$\begin{aligned}
H_-(t) &:= \sum_{1\leqslant k\leqslant 2j\leqslant n} F_{k,2j}(t) + \sum_{1\leqslant k<2j-1\leqslant n} G_{k,2j-1}(t) \\
&= \sum_{1\leqslant i\leqslant j\leqslant n/2} \big(F_{2i-1,2j}(t) + F_{2i,2j}(t) + G_{2i-1,2j+1}(t) + G_{2i,2j+1}(t)\big) \\
&= \sum_{1\leqslant i\leqslant j\leqslant n/2} \Big(f_{2i-1}(a_{2i-1}-x) * g_{2j}(1-x+2A_{2i-1,2j}-a_{2i-1}) \\
&\quad + f_{2i}(a_{2i-1}+x) * g_{2j}(1-x+2A_{2i,2j}+a_{2i-1}) \\
&\quad + f_{2i-1}(a_{2i-1}-x) * g_{2j+1}(-1+x-2A_{2i-1,2j+1}+a_{2i-1}) \\
&\quad + f_{2i}(a_{2i-1}+x) * g_{2j+1}(-1+x-2A_{2i,2j+1}-a_{2i-1})\Big) \\
&= \sum_{1\leqslant i\leqslant n/2} \Big[(f_{2i-1}(a_{2i-1}-x) + f_{2i}(a_{2i-1}+x)) \\
&\quad * \sum_{m\geqslant 2i} g_m((-1)^m(1-x+2A_{2i,m}+a_{2i-1}))\Big] \\
&= \sum_{1\leqslant i\leqslant n/2} \tilde{f}^{2i-1}(x) * G_{2i}(x).
\end{aligned}$$

By applying assertion (1) in Lemma 3.2 to relation (3.19), we complete the proof. □

In what follows, we need the following definition:

Definition 3.1. ([12,13,15]) *The determinant of the functional matrix $F(x) = (f_{ij}(x))_{i,j=1}^n$ ($f_{ij} \in L_p[0,1]$) calculated with respect to the convolution*

$$(f*g)(x) = \int_0^x f(x-t)g(t)\,dt = \int_0^x g(x-t)f(t)\,dt = (g*f)(x)$$

is called its $*$-determinant and denoted by $*$-det $F(x)$. Similarly, the $*$-minors of the matrix $F(x)$ are defined as minors calculated with respect to the convolution. The $*$-rank of the matrix $F(x)$ is defined as the maximum rank of its $*$-minors satisfying the ε-condition (1.2).

Lemma 3.3. *Suppose that $F(x) = \{f_{ij}(x)\}_{i,j=1}^{n}$ is an $n \times n$ matrix function with elements $f_{ij} \in L_p[0,1]$ and $g(x) = \{g_1(x), \dots, g_n(x)\}$ is a vector function with elements $g_i \in L_{p'}[0,1]$. Then the following assertions are true:*
(1) If $*$-rank $F(x) = n$, then the equation $F(x) * g(x) = 0$ possesses only the trivial solution $g(x) = \{0, \dots, 0\}$.
(2) If $*$-rank $F(x) < n$, then, for any $\varepsilon > 0$, there exists a nontrivial solution $g(x)$ of the equation $F(x) * g(x) = 0$ such that $\operatorname{supp} g_i \subset [1-\varepsilon, 1]$.
(3) Let $A(x) = (a_{ij}(x))_{i,j=1}^{k}$ and $B(x) = (b_{ij}(x))_{i,j=1}^{k}$ be functional matrices, let $a_{ij}, b_{ij} \in L_p[0,1]$, and let $*$-rank $A = k$. Then there exists $R > 1$ such that $*$-rank $(A + \mu B) = k$ for $\mu \in [0, \frac{1}{R}] \cup [R, \infty)$.

Proof. (1) Let $\tilde{F}(x)$ be a matrix $*$-adjoint to $F(x)$. In this case, for $F(x) * g(x) = 0$, we find

$$\begin{aligned} &\operatorname{diag}(*\text{-}\det F(x), \dots, *\text{-}\det F(x)) * (g_1(x), \dots, g_n(x)) \\ &\qquad = \tilde{F}(x) * F(x) * g(x) = (0, \dots, 0) \quad \text{for all } x \in [0,1], \end{aligned}$$

i.e., $*$-det $F(x) * g_i(x) = 0$ for all $x \in [0,1]$. Since $*$-rank $F(x) = n$, the $*$-determinant $*$-det $F(x)$ satisfies the ε-condition (1.2) and, hence, by the Titchmarsh convolution theorem, $g_i(x) = 0$ for almost all $x \in [0,1]$.

(2) (i) Let $*$-rank $F(x) = r$, $0 < r \leqslant n-1$, let $F_{1,2,\dots,\dots,r+1}^{N-r+1,\dots,N}(x)$ be an $r \times (r+1)$ submatrix of $F(x)$ located in its left bottom corner, i.e., formed by the intersection of the $N, N-1, \dots, N-r+1$st rows with the $1, 2, \dots, r+1$st columns of the matrix, and let $F_i(x)(1 \leqslant i \leqslant r+1)$ be an $r \times r$ matrix obtained from the matrix $F_{1,2,\dots,\dots,r+1}^{N-r+1,\dots N}(x)$ by deleting its ith column. Without loss of generality, we can assume that $*$-rank $F_1(x) = r$. Then, for some $\varepsilon_1 > 0$, the vector

$$\breve{g}(x) = (\breve{g}_1(x), \dots, \breve{g}_n(x)) := (*\text{-}\det F_1(x), \dots, *\text{-}\det F_{r+1}(x), 0, \dots, 0)$$

satisfies the equation

$$F(x) * \breve{g}(x) = 0, \qquad x \in [0, \varepsilon_1]. \tag{3.20}$$

We now set $\varepsilon_2 := \min(\varepsilon, \varepsilon_1)$. Then the vector

$$g(x) = (\chi_{[1-\varepsilon_2,1]}(x) * \breve{g}_1(x), \dots, \chi_{[1-\varepsilon_2,1]}(x) * \breve{g}_n(x))$$

satisfies the conditions

$$F(x) * g(x) = 0 \text{ for all } x \in [0,1],$$

$$\operatorname{supp} g_i(x) \subset [1-\varepsilon_2, 1] \subset [1-\varepsilon, 1] \quad (i = 1, \dots, n),$$

$$g_1(x) \not\equiv 0.$$

The last condition is satisfied because, by assumption, the function $\breve{g}_1(x) = *\text{-det}\, F_1(x)$ satisfies the ε-condition (1.2).

(ii) If $*\text{-rank}\, F(x) = 0$, then we set $\breve{g}(x) := (1, 0, \dots, 0)$. Hence, Eq. (3.20) is satisfied and the remaining part of the proof coincides with the proof in case (i).

(3) We consider the function $\Phi(\mu, x) := *\text{-det}(A(x) + \mu B(x))$. In this case, we have

$$\Phi(\mu, x) = \mu^m \phi_m(x) + \mu^{m-1}\phi_{m-1}(x) + \dots \mu\phi_1(x) + \phi_0(x),$$

where $\phi_m(x) \not\equiv 0$ for some $m \leqslant n$ and $\phi_0(x) = *\text{-det}\, A$. Assume that there exist $m+1$ numbers a_i and $m+1$ different numbers μ_i such that $\Phi(\mu_i, x) = 0$ for $x \in [0, a_i]$. If $\phi_0(a_i) = 0$, then, by virtue of the condition $*\text{-rank}\, A = k$, one can choose $a_i' \in [0, a_i]$ such that $\phi_0(a_i') \neq 0$. Therefore, we assume that $\phi_0(a_i) \neq 0$. Since the numbers μ_i are different, we conclude that $\phi_m(x) = \dots = \phi_0(x) = 0$ for all $x \in \bigcap [0, a_i]$, which contradicts the condition $*\text{-rank}\, A = k$. Thus, there are at most $m+1$ numbers μ satisfying the condition $*\text{-rank}\,(A + \mu B) = k$, which proves assertion (3). □

The following two theorems describe cyclic systems of vectors of the operator $V_{q,1}^{\alpha}$ for even and odd α:

Theorem 3.1. *Assume that the function $q(x)$ is equivalent to a function $d_\pi(x)$ of the form* (2.9) *and α is odd. Then*

(1) $\mu_{V_{q,1}^{\alpha}} = [\frac{n-1}{2}] + 1$*;*

(2) *a system $\{f_j\}_{j=1}^N$ of vectors f_j generates a cyclic subspace in $L_p[0,1]$ for the operator $V_{q,1}^{\alpha}$ if and only if*

(*a*) $N \geqslant [\frac{n-1}{2}] + 1$*;*

(*b*) *the matrices*

$$F_1(x) = \begin{pmatrix} \tilde{f}_1^0(x) & \tilde{f}_1^2(x) & \dots & \tilde{f}_1^{2i}(x) & \dots & \tilde{f}_1^{2[(n-1)/2]}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{f}_j^0(x) & \tilde{f}_j^2(x) & \dots & \tilde{f}_j^{2i}(x) & \dots & \tilde{f}_j^{2[(n-1)/2]}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{f}_N^0(x) & \tilde{f}_N^2(x) & \dots & \tilde{f}_N^{2i}(x) & \dots & \tilde{f}_N^{2[(n-1)/2]}(x) \end{pmatrix}$$

and

$$F_2(x) = \begin{pmatrix} \tilde f_1^1(x) & \tilde f_1^3(x) & \dots & \tilde f_1^{2i-1}(x) & \dots & \tilde f_1^{2[n/2]-1}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde f_j^1(x) & \tilde f_j^3(x) & \dots & \tilde f_j^{2i-1}(x) & \dots & \tilde f_j^{2[n/2]-1}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde f_N^1(x) & \tilde f_N^3(x) & \dots & \tilde f_N^{2i-1}(x) & \dots & \tilde f_N^{2[n/2]-1}(x) \end{pmatrix}$$

have the maximum $*$-rank, i.e.,

$$*\text{-rank}\, F_1(x) + *\text{-rank}\, F_2(x) = n; \tag{3.21}$$

(3) $\operatorname{disc} V_{q,1}^{\alpha} = [\frac{n-1}{2}] + 1$.

Proof. (1)–(2) By using the equality $A_{2i,m} - A_{2i-1,m} = -a_{2i-1}$ and the definitions of the functions G_m (3.9)–(3.10), we find

$$G_{2i}(1 - x + a_{2i-1}) = g_{2i}(x) + \sum_{m \geqslant 2i+1} g_m((-1)^m(x + 2A_{2i,m})),$$

$$G_{2i+1}(1 + x - a_{2i}) = \sum_{m \geqslant 2i+1} g_m((-1)^m(x + 2A_{2i,m})),$$

$$\begin{aligned} G_{2i-1}(1 - x + a_{2i-2}) &= g_{2i-1}(x) + \sum_{m \geqslant 2i} g_m((-1)^{m-1}(x - 2A_{2i-1,m})) \\ &= g_{2i-1}(x) + \sum_{m \geqslant 2i} g_m((-1)^{m-1}(x - 2A_{2i,m} - 2a_{2i-1})), \end{aligned}$$

$$G_{2i}(1 + x - a_{2i-1}) = \sum_{m \geqslant 2i} g_m((-1)^{m-1}(x - 2A_{2i,m} - 2a_{2i-1}))$$

and, therefore,

$$g_{2i}(x) = G_{2i}(1 - x + a_{2i-1}) - G_{2i+1}(1 + x - a_{2i}), \tag{3.22}$$

$$g_{2i-1}(x) = G_{2i-1}(1 - x + a_{2i-2}) - G_{2i}(1 + x - a_{2i-1}). \tag{3.23}$$

Let

$$E_{V_{q,1}^{\alpha}}(f_1, \dots, f_N) := \operatorname{span}\{V_{q,1}^{\alpha n} f_i : 1 \leqslant i \leqslant N, n \geqslant 0\}.$$

Then, by virtue of Proposition 3.1, $g(x) = (g_1(x), \dots, g_n(x)) \in E_{V_{q,1}^{\alpha}}(f_1, \dots, f_N)^{\perp}$ if and only if

$$F_1(x) * \breve G_1(x) = 0 \text{ and } F_2(x) * \breve G_2(x) = 0, \tag{3.24}$$

where $\breve{G}_1(x) = (G_1(x), \dots, G_{2[(n-1)/2]+1}(x))$, $\breve{G}_2(x) = (G_2(x), \dots, G_{2[n/2]}(x))$, and the functions G_m are constructed from g by using relations (3.9)–(3.10). If condition (3.21) is satisfied, then, according to assertion (1) in Lemma 3.3, Eqs. (3.24) possess only the trivial solutions $\breve{G}_1(x)$ and $\breve{G}_2(x)$, i.e., $G_i \equiv 0$ $(i = 1, \dots, n)$ and, hence, in view of (3.22)–(3.23), $g_i \equiv 0$ $(i = 1, \dots, n)$, which proves that condition (3.21) is sufficient.

Conversely, if condition (3.21) is not satisfied, then, according to assertion (2) in Lemma 3.3, there exist functions $G_m(x)$ $(m = 1, \dots, n)$ such that at least one of them is not identically equal to zero and

$$\operatorname{supp} G_{2i+1} \subset [1 + a_{2i-1} - a_{2i}, 1] \bigcap [1 + a_{2i} - a_{2i+1}, 1],$$

$$\operatorname{supp} G_{2i} \subset [1 + a_{2i-1} - a_{2i}, 1] \bigcap [1 + a_{2i} - a_{2i+1}, 1].$$

Then the functions $g_m(x)$ constructed by using relations (3.22)–(3.23) are such that $\operatorname{supp} g_m(x) \subset [a_{m-1}, a_m]$ and at least one of the functions g_m is not identically equal to zero. Therefore, $g \not\equiv 0$ and, hence, $g(x) = +_{m=1}^{n} g_m(x) \in E_{V_{q,1}^{\alpha}}(f_1, \dots, f_N)^{\perp}$.

(3) We now consider the case of even n. Let $E = \operatorname{span}\{f_1, \dots, f_N\}$ be an N-dimensional subspace cyclic for the operator $V_{q,1}^{\alpha}$. It is necessary to show that this subspace contains an $n/2$-dimensional subspace which is also cyclic for the operator $V_{q,1}^{\alpha}$. Since $E \in \operatorname{Cyc} V_{q,1}^{\alpha}$, there exist $n/2 \times n/2$ square submatrices $A_1(x)$ and $A_2(x)$ of the matrices $F_1(x)$ and $F_2(x)$ such that $*\text{-rank}\, A_1(x) = n/2$ and $*\text{-rank}\, A_2(x) = n/2$. Assume that the matrices $A_1(x)$ and $A_2(x)$ are constructed from the vectors $f_{i_1}, \dots, f_{i_{n/2}}$ and $f_{j_1}, \dots, f_{j_{n/2}}$, respectively. Then the n-dimensional subspace $E_\varepsilon := \operatorname{span}\{f_{i_1} + \varepsilon f_{j_1}, \dots, f_{i_{n/2}} + \varepsilon f_{j_{n/2}}\}$ is cyclic for some $\varepsilon > 0$. Indeed, the matrices $F_1(x)$ and $F_2(x)$ constructed for the system of vectors $f_{i_1} + \varepsilon f_{j_1}, \dots, f_{i_{n/2}} + \varepsilon f_{j_{n/2}}$ have the form

$$F_{1\varepsilon}(x) = A_1(x) + \varepsilon B_1(x),$$

$$F_{2\varepsilon}(x) = \varepsilon A_2(x) + B_2(x) = \varepsilon(A_2(x) + 1/\varepsilon B_2(x)),$$

where $B_1(x)$ and $B_2(x)$ are certain matrices. Hence, in view of assertion (3) in Lemma 3.3, we have $*\text{-rank}\, F_{1\varepsilon}(x) = n/2$ and $*\text{-rank}\, F_{2\varepsilon}(x) = n/2$ for some $\varepsilon > 0$, i.e., the subspace E_ε is cyclic for the operator $V_{q,1}^{\alpha}$.

The case of odd n is proved similarly. □

The following theorem is proved in exactly the same way as Theorem 3.1:

Theorem 3.2. *Assume that the function $q(x)$ is equivalent to a function $d_\pi(x)$ of the form* (2.9) *and α is even. Then*

(1) $\mu_{V^\alpha_{q,1}} = n$;

(2) *the system $\{f_j\}_{j=1}^N$ of vectors f_j generates a cyclic subspace in $L_p[0,1]$ for the operator $V^\alpha_{q,1}$ if and only if*

(*a*) $N \geqslant n$;

(*b*) *the matrix*

$$F(x) = \begin{pmatrix} \tilde f_1^0(x) & \tilde f_1^1(x) & \dots & \tilde f_1^i(x) & \dots & \tilde f_1^{n-1}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde f_j^0(x) & \tilde f_j^1(x) & \dots & \tilde f_j^i(x) & \dots & \tilde f_j^{n-1}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde f_N^0(x) & \tilde f_N^1(x) & \dots & \tilde f_N^i(x) & \dots & \tilde f_N^{n-1}(x) \end{pmatrix}$$

has the maximum $*$-rank, i.e., $*\text{-rank}\, F(x) = n$;

(3) $\operatorname{disc} V^\alpha_{q,1} = n$.

Corollary 3.1. *Suppose that the function $q(x)$ is equivalent to a function $d_\pi(x)$ of the form* (2.9). *Then the operator $V^\alpha_{q,1}$ is cyclic if and only if $n = 1$ or $n = 2$ and α is odd. Furthermore,*

(1) *if $n = 1$, then $V^\alpha_{q,1} = J^\alpha$ and*

$$f(x) \in \operatorname{Cyc} J^\alpha \Leftrightarrow 0 \in \operatorname{supp} f(x);$$

(2) *if $n = 2$, then $q(x) = \chi_{[0,a]}(x) - \chi_{[a,1]}(x)$ and, for odd α,*

$$f(x) \in \operatorname{Cyc} V^\alpha_{q,1} \Leftrightarrow 0 \in \operatorname{supp} f(x) * (f(a-x) + f(a+x)).$$

Example 3.1. Let $\pi = \{0 < a < b < 1\}$. Then $n = 3$ and $q(x) = 1 - 2\chi_{[a,b]}(x)$. Hence, by virtue of Theorems 3.1 and 3.2,

$$\mu := \mu_{V^\alpha_{q,1}} = \begin{cases} [\frac{n-1}{2}] + 1 = 2, & \alpha \text{ is odd}, \\ n = 3, & \alpha \text{ is even}. \end{cases}$$

The subspace $E := \operatorname{span}\{f_1, \dots, f_\mu\}$ is cyclic for the operator $V^\alpha_{q,1}$ if and only if

(1) for odd α, the following conditions are satisfied:

$$*\text{-rank} \begin{pmatrix} f_1(x) & f_1(b-x) + f_1(b+x) \\ f_2(x) & f_2(b-x) + f_2(b+x) \end{pmatrix} = 2,$$

$$*\text{-rank} \begin{pmatrix} f_1(a-x) + f_1(a+x) \\ f_2(a-x) + f_2(a+x) \end{pmatrix} = 1.$$

(2) for even α, we have $*$-rank $F(x) = 3$, where

$$F(x) = \begin{pmatrix} f_1(x) & f_1(a-x)+f_1(a+x) & f_1(b-x)+f_1(b+x) \\ f_2(x) & f_2(a-x)+f_2(a+x) & f_2(b-x)+f_2(b+x) \\ f_3(x) & f_3(a-x)+f_3(a+x) & f_3(b-x)+f_3(b+x) \end{pmatrix}.$$

Example 3.2. In the previous example, we set $f_1(x) := \chi_{[0,a]}(x)$, $f_2(x) := \chi_{[a,b]}(x)$, and $f_3(x) := \chi_{[b,1]}(x)$. Then, for $x < \varepsilon := \min\{a, b-a, 1-b\}$, we get

$$F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, $*$-det $F(x) = 1 * 1 * 1 = x^3/6$ for $x \in [0, \varepsilon]$ and, hence, $*$-rank $F(x) = 3$. Thus, the subspace $E = \operatorname{span}\{f_1, f_2, f_3\}$ is cyclic for the operator $V_{q,1}^2$.

If a function $f(x)$ is continuous at zero and $f(0) \neq 0$, then it is clear that $f(x)$ satisfies the ε-condition (1.2). This means that the function $f(x)$ is cyclic for the operator J^α. A similar effect is also observed for the operator $V_{q,w}^\alpha$.

Corollary 3.2. *Let $N = \mu_{V_{q,1}^\alpha}$ and let functions $f_i(x) \in L_p[0,1]$ $(i = 1, \dots, N)$ be continuous at points a_i $(i = 0, \dots, n-1)$. In order that the system $\{f_i\}_{i=1}^N$ of vectors f_i generate a cyclic subspace in $L_p[0,1]$ for the operator $V_{q,1}^\alpha$, it is sufficient that the following conditions be true for odd α:*

$$\det \begin{pmatrix} f_1(0) & f_1(a_2) & \dots & f_1(a_{2[(n-1)/2]}) \\ \vdots & \vdots & \vdots & \vdots \\ f_{[\frac{n-1}{2}]+1}(0) & f_{[\frac{n-1}{2}]+1}(a_2) & \dots & f_{[\frac{n-1}{2}]+1}(a_{2[(n-1)/2]}) \end{pmatrix} \neq 0, \tag{3.25}$$

$$\det \begin{pmatrix} f_1(a_1) & f_1(a_3) & \dots & f_1(a_{2[n/2]-1}) \\ \vdots & \vdots & \vdots & \vdots \\ f_{[\frac{n-1}{2}]+1}(a_1) & f_{[\frac{n-1}{2}]+1}(a_3) & \dots & f_{[\frac{n-1}{2}]+1}(a_{2[n/2]-1}) \end{pmatrix} \neq 0, \tag{3.26}$$

and the following inequality hold for even α:

$$\det \begin{pmatrix} f_1(0) & f_1(a_1) & \dots & f_1(a_{n-1}) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(0) & f_n(a_1) & \dots & f_n(a_{n-1}) \end{pmatrix} \neq 0. \tag{3.27}$$

Proof. We consider the case of even α. In this case, $N = n$. Let us show that if condition (3.27) is satisfied, then $*$-rank $F(x) = n$ and, hence, the system $\{f_i\}_{i=1}^n$ generates a cyclic subspace. Indeed, we have $*\text{-}\det F(x) = \int_0^x \breve{F}(x,t)dt$. Since the functions f_i are continuous at the points a_i $(i = 0, \ldots, n-1)$, the function $\breve{F}(x,t)$ is continuous in the neighborhood of zero. Further, since $\breve{F}(0,0) = \det F(0) \neq 0$, the function $\breve{F}(x,t)$ does not vanish in some neighborhood of zero and, hence, the function $*\text{-}\det F(x) = \int_0^x \breve{F}(x,t)dt$ satisfies the ε-condition (1.2).

The case of odd α is proved similarly. □

Remark 3.2. It is easy to see that conditions (3.25)–(3.26) are not necessary even for $n = 1$.

As a corollary of Theorems 3.1 and 3.2, we present the following Malamud's result [12, 13] concerning cyclic subspaces of the operator $A = J \otimes B$ (in [12, 13], a similar result was established for more general operators of the form $A = J^\alpha \otimes B,\ \alpha > 0$):

Proposition 3.2. (see [12,13]) *Let $B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ be a diagonal matrix with numbers $\lambda_i > 0$ and let $A = J^\alpha \otimes B = \bigoplus_{i=1}^n \lambda_i J^\alpha$ be an operator acting in the space $L_p[0,1] \otimes \mathbb{C}^n$. Then*

(1) *$\mu_A = n$;*

(2) *the system of vectors*

$$h^j = \{h_1^j, \ldots, h_n^j\} \in L_p[0,1] \otimes \mathbb{C}^n \qquad (1 \le j \le N)$$

generates a cyclic subspace for the operator A if and only if

1. *$N \ge n$;*
2. *the matrix*

$$H(x) = \begin{pmatrix} h_1^1(\lambda_1^{-1/\alpha}x) & h_2^1(\lambda_2^{-1/\alpha}x) & \ldots & h_n^1(\lambda_n^{-1/\alpha}x) \\ \ldots & \ldots & \ldots & \ldots \\ h_1^N(\lambda_1^{-1/\alpha}x) & h_2^N(\lambda_2^{-1/\alpha}x) & \ldots & h_n^N(\lambda_n^{-1/\alpha}x) \end{pmatrix}$$

has the maximum $*$-rank, i.e,

$$*\text{-}\operatorname{rank} H(x) = n. \tag{3.28}$$

Proof. We present the proof for the case $\alpha = 1$. It is possible to assume that $\sum_{i=1}^n \lambda_i = \frac{1}{2}$. Thus, we set

$$a_{2i} := 2\sum_{l=1}^i \lambda_l,\ a_{2i-1} := a_{2i-2} + \lambda_i = a_{2i} - \lambda_i \qquad (i = 1, \ldots, n).$$

Further, by using the points a_i, we construct the function $q(x)$ [see(2.9)] and set $q_i(x) := \chi_{[a_{2i-2},a_{2i}]}(x)q(x)$. Then

$$E_i := E_{V_{q,1}}(q_i) := \operatorname{span}\{(V_{q,1}^n q_i)(x) : \ n \geqslant 0\} \in \operatorname{Lat} V_{q,1}.$$

So $E_i \subset \chi_{[a_{2i-2},a_{2i}]}(x)L_p[0,1]$, since $\operatorname{supp}(V_{q,1}^n q_i)(x) = [a_{2i-2}, a_{2i}]$. Now let $E = \dot{+}E_i$ and let $f(x) = \bigoplus_{k=1}^{2n} f_k(x) \in E$, where, as in (3.11),

$$f_k(x) = \chi_{[a_{k-1},a_k]}(x)f(x) \qquad (k = 1, \ldots, 2n).$$

Thus, in view of the construction of the space E, we have

$$f_{2i-1}(a_{2i-2} + x) = -f_{2i}(a_{2i} - x) \qquad (i = 1, \ldots, n).$$

Assume that the operator $P : E \to L_p[0,1] \otimes \mathbb{C}^n$ acts according to the rule

$$(Pf)(x) = (P\bigoplus_{j=1}^{2n} f_j)(x) = (\bigoplus_{i=1}^{n} P(f_{2i-1} \bigoplus f_{2i}))(x)$$
$$= \bigoplus_{i=1}^{n}(f_{2i-1}(a_{2i-2} + \lambda_i x) := (h_1(x), \ldots, h_n(x)) := h(x), \ x \in [0,1].$$

Then the operator P is bounded together with its inverse. The indicated inverse operator P^{-1} has the form

$$f(x) := (P^{-1}h)(x) = \begin{cases} h(\lambda_i^{-1}(x - a_{2i-2})), & x \in [a_{2i-2}, a_{2i-1}], \ 1 \leqslant i \leqslant n, \\ -h(\lambda_i^{-1}(a_{2i} - x)), & x \in [a_{2i-1}, a_{2i}], \quad 1 \leqslant i \leqslant n \end{cases}$$

and, in particular,

$$f_{2i-1}(x) = \begin{cases} h(\lambda_i^{-1}(x - a_{2i-2})), & x \in [a_{2i-2}, a_{2i-1}], \\ 0, & x \notin [a_{2i-2}, a_{2i-1}]. \end{cases} \tag{3.29}$$

Since $E_i \in \operatorname{Lat} V_{q,1}^{\alpha}$, the equalities

$$(PV_{q,1}f)(x) = \bigoplus_{i=1}^{n} q(a_{2i-2} + \lambda_i x) \int_0^{a_{2i-2}+\lambda_i x} f_{2i-1}(t)\,dt$$
$$= \bigoplus_{i=1}^{n} \int_{a_{2i-2}}^{a_{2i-2}+\lambda_i x} f_{2i-1}(t)\,dt = \lambda_i \int_0^x f_{2i-1}(a_{2i-2} + \lambda_i t)\,dt = (APf)(x).$$

hold for all $f \in E$, i.e., the operators $V_{q,1}\restriction_E$ and A are similar. Therefore, $H \in \operatorname{Cyc} \bigoplus_1^n \lambda_i J$ if and only if $P^{-1}H \in \operatorname{Cyc} V_{q,1}\restriction_E$. Hence, the system of vectors $h^j = \{h_1^j, \dots, h_n^j\} \in L_p[0,1] \otimes \mathbb{C}^n$ $(1 \le j \le N)$ generates a cyclic subspace for the operator A if and only if the system of vectors $P^{-1}h^j = \{P_1^{-1}h_1^j, \dots, P_n^{-1}h_n^j\} := \{f_1^j, f_2^j \dots, f_{2n-1}^j, f_{2n}^j\} := f^j$ $(1 \le j \le N)$ generates a cyclic subspace for the operator $V_{q,1}\restriction_E$. We set

$$F = \operatorname{span}\{(V_{q,1}^n f^j)(x) : 1 \leqslant j \leqslant N, n \geqslant 0\}.$$

By virtue of Proposition 3.2, $g = \{g_1, \dots, g_{2n}\} \in F^\perp$ if and only if conditions (3.7) and (3.8) are satisfied for all $f \in H$. Since, for $1 \le j \le N$ and $1 \le i \le n$, we have

$$f_{2i-1}^j(a_{2i-1} - x) + f_{2i}^j(a_{2i} + x) = 0, \tag{3.30}$$

$$g_{2i-1}(a_{2i-1} - x) + g_{2i}(a_{2i} + x) = 0,$$

$$G_{2i-1}(x) = g_{2i-1}(1 - x - 2A_{2i-1,2i-1} + a_{2i-2}) = g_{2i-1}(1 - x + a_{2i-2}),$$

$$G_{2i}(x) = 0,$$

conditions (3.7) and (3.8) take the form

$$\sum_{i=1}^n (f_{2i-2}^j(a_{2i-2} - x) + f_{2i-1}^j(a_{2i-2} + x)) * g_{2i-1}(1 - x + a_{2i-2}) = 0 \qquad (j = 1, \dots, N),$$

$$0 = 0$$

or, by virtue of (3.30),

$$f_1^j(x) * g_1(1 - x) + \sum_{i=2}^n (f_{2i-3}^j(a_{2i-4} + x) + f_{2i-1}^j(a_{2i-2} + x)) * g_{2i-1}(1 - x + a_{2i-2}) = 0.$$

By Lemma 3.3, $g(x) \equiv 0$ if and only if $*\text{-rank}\, F(x) = n$, where

$$F(x) = \begin{pmatrix} f_1^1(x) & \dots & f_{2n-3}^1(a_{2n-4} + x) + f_{2n-1}^1(a_{2n-2} + x) \\ \vdots & \vdots & \vdots \\ f_1^N(x) & \dots & f_{2n-3}^N(a_{2n-4} + x) + f_{2n-1}^N(a_{2n-2} + x) \end{pmatrix}.$$

It is easy to see that $*\text{-rank}\, F(x) = *\text{-rank}\, \breve{F}(x)$, where

$$\breve{F}(x) = \begin{pmatrix} f_1^1(x) & \dots & f_{2i-1}^1(a_{2i-2} + x) & f_{2n-1}^1(a_{2n-2} + x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^N(x) & \dots & f_{2i-1}^N(a_{2i-2} + x) & f_{2n-1}^N(a_{2n-2} + x) \end{pmatrix}.$$

By virtue of (3.29), we have $f^j_{2i-1}(a_{2i-2}+x) = h^j_i(\lambda_i^{-1}x)$ and, hence, $\breve{F}(x) = H(x)$, which completes the proof of the proposition. □

4. Cyclic Subspaces. Case II

In this section, we prove that the operator $V^{\alpha}_{q,1}$ does not have finite-dimensional cyclic subspaces if the function q takes the values ± 1 and is not equivalent to functions $d_\pi(x)$ of the form (2.9).

Let J_{ab} be an operator of integration J acting in the space $L_p[a,b]$ (here, a is not necessarily smaller than b) according to the rule $(J_{ab}f)(x) = \int_a^x f(t)\,dt$. Thus, if, e.g., $p = 2$, then $J_{01} = J$ and $J_{10} = -J^*$.

Lemma 4.1. *The operators J_{ab} and $(b-a)J$ are similar.*

Proof. Consider an operator $T_{ab} : L_p[a,b] \to L_p[0,1]$, $(T_{ab}f)(x) = f(\frac{x-a}{b-a})$. In this case,

$$J_{ab}T_{ab} = \int\limits_a^x f(\frac{t-a}{b-a})\,dt = (b-a)\int\limits_0^{\frac{x-a}{b-a}} f(t)\,dt = T_{ab}(b-a)J.$$

Since the operator T_{ab} is invertible: $(T_{ab}^{-1}f)(x) = f(x(b-a)+a)$, the fact that the operators J_{ab} and $(b-a)J$ are similar is proved. □

If $0 \in [a,b]$, then by J_{ab0} we denote an operator acting in $L_p[a,b]$ (where a is not necessarily smaller than b) according to the rule $(J_{ab0}f)(x) = \int_0^x f(t)\,dt$. If $a = 0$ or $b = 0$, then $J_{ab0} = J_{ab}$. The operator J_{ab0} decomposes into the direct sum $J_{ab0} = J_{0a} \oplus J_{0b}$ and, by virtue of Lemma 4.1, is similar to the operator $(a-0)J \oplus (b-0)J = aJ \oplus bJ$. Note that, in view of the inequality $ab < 0$, Lemma 3.2 implies that the last operator is cyclic.

Lemma 4.2. *Let $|q(x)| = 1$ and let*

$$Q(x) := \int\limits_0^x q(t)\,dt \quad \textit{for almost all } x \in [0,1].$$

We set $m := \min\limits_{x\in[0,1]} Q(x)$, $M := \max\limits_{x\in[0,1]} Q(x)$, and $\lambda(x) := \mu\{t : Q(t) < x\}$, $x \in [m,M]$. Then $\lambda'(x) \geqslant 1$.

Proof. Assume that

$E_+ = \{t :$ there exist $Q'(t)$ and $Q'(t) = 1\}$,

$E_- = \{t :$ there exist $Q'(t)$ and $Q'(t) = -1\}$.

Since $Q \in AC[0,1]$, we have $\mu\{E_+ \cup E_-\} = 1$ and $\mu\{Q(E_+ \cup E_-)\} = M - m$. Let us show that, for any $x \in Q(E_+ \cup E_-)$, one can find a sequence $\{\Delta x_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} \Delta x_i = 0, \qquad \lambda(x + \Delta x_i) - \lambda(x) > \Delta x_i.$$

Since the derivative of $\lambda(x)$ exists almost everywhere, this means that $\lambda'(x) \geqslant 1$.

(i) Let $x = Q(t)$, $t \in E_+$. Then there exists $\delta_1 > 0$ such that $Q(t + \Delta t) - Q(t) > 0$ for all $\Delta t \in [0, \delta_1]$. Further, let t_1 be a point of maximum of the function $Q(s)$ in the segment $[t, t + \delta_1]$, i.e., $Q(t_1) = \max\limits_{s\in[t,t+\delta_1]} Q(s)$. We set $\Delta x_1 := Q(t_1) - Q(t) = \int_t^{t_1} q(s)\, ds$. Then

$$0 < \Delta x_1 < \int\limits_t^{t_1} |q(s)|\, ds = t_1 - t < \delta_1$$

and $Q(t_1) = Q(t) + \Delta x_1 = x + \Delta x_1$. Hence, we get

$$\begin{aligned}\lambda(x + \Delta x_1) - \lambda(x) &= \mu\{s : x < Q(s) < x + \Delta x_1\} \\ &= \mu\{s : Q(t) < Q(s) < Q(t_1)\} > t_1 - t > \Delta x_1,\end{aligned}$$

and, moreover, $\Delta x_1 < \delta_1$. By setting $\delta_i := 1/2\delta_{i-1}$ $(i \geqslant 2)$, after similar transformations, we conclude that $\lambda(x+\Delta x_i)-\lambda(x) > \Delta x_i$, where $\Delta x_i < \delta_i = = (1/2)^{i-1}\delta_1 \to 0$.

(ii) The case where $x = Q(t)$, $t \in E_-$, is studied in exactly the same way as (i). □

Proposition 4.1. *Let*

$$E_q := \operatorname{span}\{(V_{q,1}^n q)(x) : n \geqslant 1\} = \operatorname{span}\{\frac{Q(x)^n}{n!} q(x) : n \geqslant 1\}.$$

Then

(1) $E_q \in \operatorname{Lat} V_{q,1}$;

(2) *The operator* $V_{q,1} \restriction_{E_q}$ *is quasisimilar to the operator* $(J_{mM0} f)(x) := \int_0^x f(t)dt$ *acting in* $L_p[m, M]$, *where* $m := \min\limits_{x\in[0,1]} Q(x) \leqslant 0 \leqslant M := \max\limits_{x\in[0,1]} Q(x)$.

Proof. Assume that

$$(X_1 f)(x) = q(x) \int\limits_0^{Q(x)} f(t)\, dt, \qquad X_1 \ : L_p[m, M] \to L_p[0, 1].$$

Then

$$\|X_1 f\|^p_{L_p[0,1]} = \int_0^1 |q(x)|^p \left| \int_0^{Q(x)} f(t)\,dt \right|^p dx$$

$$\leq \int_0^1 \left(\int_m^M |f(t)|\,dt \right)^p dx \leq (M-m)^{\frac{p}{p'}} \|f\|^p$$

and, hence, the operator X_1 is bounded. If $f \in ker X_1$, then $(X_1 f)(x) = \int_0^{Q(x)} f(t)dt = 0$ for almost all $x \in [0,1]$. Therefore, $\int_0^x f(t)dt = 0$ for almost all $x \in [m,M]$ and, consequently, $ker X_1 = \{0\}$.

Since

$$\overline{\mathfrak{R}(X_1)} = \text{span}\{X_1 \frac{x^n}{n!} : n \geqslant 0\} = \text{span}\{q(x)\frac{Q(x)^n}{n!} : n \geqslant 1\} = E_q,$$

the operator $X := X_1 \ : L_p[m,M] \to E_q$ is a deformation. The relation $XJ_{mM0} = V_1 X$ is true in the set of polynomials

$$XJ_{mM0}\frac{x^n}{n!} = q(x) \int_0^{Q(x)} \int_0^t \frac{s^n}{n!}\,ds\,dt q(x) \int_0^{Q(x)} \frac{s^{n+1}}{(n+1)!}\,ds =$$

$$= q(x)\frac{Q(x)^{n+2}}{(n+2)!} = q(x) \int_0^x \frac{Q(t)^{n+1}}{(n+1)!}\,dQ(t)$$

$$= q(x) \int_0^x q(t) \int_0^{Q(t)} \frac{s^n}{n!}\,ds\,dt = V_1 X \frac{x^n}{n!}$$

and, therefore, $XJ_{mM0}f = V_1 X f$ for any function $f \in L_p[m,M]$.

We now consider an operator $Y : E_q \to L_p[m,M]$ given by the formula $Y(q(x)P_n(Q(x)) = P_n(x)$ in a set $E_{q,0}$ (dense in E_q) of polynomials $Q(x)$ equal to zero at the origin:

$$E_{q,0} = \{q(x)P_n(Q(x)) : P_n(0) = 0\} = X\{P_n(x) : P_n(0) = 0\}$$

Thus, we get

$$\int_0^1 |q(x)P_n(Q(x))|^p\,dx = \int_0^1 |P_n(Q(x))|^p\,dx = \int_m^M |P_n(x)|^p\,d\lambda(x),$$

where $\lambda(x) := mes\{t : Q(t) < x\}$, $x \in [m, M]$. By virtue of Lemma 4.2,

$$\int_m^M |P_n(x)|^p \, d\lambda(x) \geq \int_m^M |P_n(x)|^p \, dx.$$

Hence, the operator Y is bounded in an everywhere dense set and can be extended as bounded operator onto the entire E_q. Further, $XY = V_{q,1} \restriction_{E_q}$ in the set $E_{q,0}$ dense in E_q and, hence, in the entire E_q. Therefore, $\ker Y = 0$. Since

$$\mathfrak{R}(Y) \supset \operatorname{span}\{Y(q(x)\frac{Q(x)^n}{n!}) : n \geqslant 1\} = \operatorname{span}\{x^n : n \geqslant 1\} = L_p[0,1],$$

the operator Y is a deformation. The operators $YV_{q,1} \restriction_{E_q}$ and $J_{mM0}Y$ also coincide in the set $E_{q,0}$ and, thus, in the entire E_q. Therefore, the operators $V_{q,1} \restriction_{E_q}$ and J_{mM0} are quasisimilar. □

Corollary 4.1. *Assume that numbers $a, b \in [0,1]$ are such that $\int_a^b q(s)ds = 0$ and, in addition, $\breve{q} := \chi_{[a,b]}(x)q(x)$ and $E_{\breve{q}} := \operatorname{span}\{(V_{q,1}^n \breve{q})(x) : n \geqslant 1\}$. Then*
(1) $E_{\breve{q}} \in \operatorname{Lat} V_{q,1} \bigcap \chi_{[a,b]} L_p[0,1]$;
(2) The operator $V_{q,1} \restriction_{E_{\breve{q}}}$ is quasisimilar to the operator $(J_{mM0}f)(x) := \int_0^x f(t)dt$ acting in $L_p[m, M]$, where

$$m := \min_{x\in[a,b]} \int_a^x q(s)\,ds \leqslant 0 \leqslant M := \max_{x\in[a,b]} \int_a^x q(s)\,ds.$$

Proof. (1) It is clear that $E_{\breve{q}} \in \operatorname{Lat} V_{q,1}$. We now prove that $E_{\breve{q}} \subset \chi_{[a,b]} L_p[0,1]$. We have

$$(V_{q,1}^{n+1}\breve{q})(x) = \begin{cases} 0, & 0 < t < a, \\ q(x)\int\limits_a^x q(t)(\int\limits_t^x q(s)\,ds)^n/n!\,dt, & a < t < b, \\ q(x)\int\limits_a^b q(t)(\int\limits_t^b q(s)\,ds)^n/n!\,dt, & b < t < 1 \end{cases}$$

$$= \begin{cases} 0, & 0 < t < a, \\ -q(x)\left[\left(\int\limits_t^x q(s)\,ds\right)^{n+1}/(n+1)!\right]\Bigg|_a^x, & a < t < b, \\ -q(x)\left[\left(\int\limits_t^b q(s)\,ds\right)^{n+1}/(n+1)!\right]\Bigg|_a^b & ,b < t < 1 \end{cases}$$

$$= \begin{cases} 0, & t \in [0,1]\backslash[a,b], \\ q(x)\Big(\int\limits_a^x q(s)\,ds\Big)^{n+1}/(n+1)!, & a < t < b. \end{cases}$$

(2) Let us show that it is possible to take $a = 0$ and $b = 1$. To this end, we set

$$q_1(x) = \breve{q}((b-a)x + a) \in L_p[0,1], \quad E_1 := \operatorname{span}\{(V^n_{q_1,1}q_1)(x) : n \geqslant 1\}$$

and show that the operators $V_{q,1}\restriction_{E_{\breve{q}}}$ and $(b-a)V_{q_1,1}\restriction_{E_1}$ are similar. After this, by using Proposition 4.3, we compete the proof of the corollary.

Let $(Tf)(x) = f((b-a)x+a)$ be an operator from E onto TE. Since, for $n \geqslant 1$,

$$\begin{aligned}(TV^n_{q,1}\breve{q})(x) &= \frac{\breve{q}((b-a)x+a)}{n!}\Bigg(\int\limits_a^{(b-a)x+a}\breve{q}(s)\,ds\Bigg)^n \\ &= \frac{q_1(x)}{n!}\Bigg((b-a)\int\limits_0^x \breve{q}((b-a)s+a)\,ds\Bigg)^n \\ &= (b-a)^n(V^n_{q_1,1}q_1)(x) = (b-a)^n(V^n_{q_1,1}T\breve{q})(x),\end{aligned}$$

we conclude that $TE = E_1$. The last equality can be rewritten in the form

$$TV_{q,1}(V^{n-1}_{q,1}\breve{q}) = (b-a)V_{q_1,1}(b-a)^{n-1}V^{n-1}_{q_1,1}T\breve{q} = (b-a)V_{q_1,1}T(V^{n-1}_{q,1}\breve{q}), \quad n \geqslant 2.$$

Thus, the operators $TV_{q,1}$ and $(b-a)V_{q_1,1}T$ coincide in a set everywhere dense in E_q and, hence, $TV_{q,1} = (b-a)V_{q_1,1}T$. Since the operator T^{-1} defined in the entire E_1 by the equality $(T^{-1}f)(x) = f(\frac{x-a}{b-a})$ is bounded, the operators $V_{q,1}\restriction_E$ and $(b-a)V_{q,1}\restriction_{E_1}$ are similar. Therefore, under the conditions of the corollary, we can set $a = 0$, $b = 1$, $\breve{q} = q$, $E_{\breve{q}} = E_q$, and $m = \min\limits_{x\in[0,1]}\int_0^x q(s)\,ds \leqslant 0 \leqslant M = \max\limits_{x\in[0,1]}\int_0^x q(s)\,ds$. □

Corollary 4.2. *Let*

$$0 \leqslant a_1 < b_1 < \cdots < a_n < b_n \leqslant 1, \ \textit{and} \int\limits_{a_i}^{b_i} q(s)\,ds = 0, \quad (i = 1,\dots,n),$$

and let

$$q_i := \chi_{[a_i,b_i]}(x)q(x), \quad E_{q_i} := \operatorname{span}\{(V^k_{q,1}q_i)(x) : k \geqslant 1\} \quad (i = 1,\dots,n).$$

Then

(1) $E := \bigoplus_{i=1}^{n} E_{q_i} \in \mathrm{Lat} V_{q,1}$;

(2) *the operator* $V_{q,1} \upharpoonright_E$ *is quasisimilar to the operator* $\bigoplus_{i=1}^{n} J_{m_i M_i 0}$, *where*

$$m_i := \min_{x\in[a_i,b_i]} \int_{a_i}^{x} q(s)\,ds \leqslant 0 \le M_i := \max_{x\in[a_i,b_i]} \int_{a_i}^{x} q(s)\,ds \quad (i = 1,\dots,n).$$

Proof. Assertions (1) and (2) follow from the assertions (1) and (2) in Corollary 4.1, respectively. □

Proposition 4.2. *Suppose that* $|q(x)| \equiv 1$ *and the function* $q(x)$ *is not equivalent to a function* $d_\pi(x)$ *(or* $-d_\pi(x)$ *) of the form* (2.9) *for any finite partition* π *of the form* (2.8). *Then* $\mu_{V_{q,1}} = \infty$ *and, therefore,* $\mu_{V_{q,1}^\alpha} = \infty$.

Proof. By $V_{q,1,r,s}$ we denote an operator acting from L_r into L_s according to the same rule as the operator $V_{q,1}$. It is clear that the operator $V_{q,1,r,s}$ is a deformation and

$$V_{q,1}V_{q,1,2,p} = V_{q,1,2,p}V_{q,1,2,2}, \qquad V_{q,1,2,2}V_{q,1,p,2} = V_{q,1,p,2}V_{q,1}.$$

This means that the operators $V_{q,1}(= V_{q,1,p,p})$ and $V_{q,1,2,2}$ are quasisimilar and, hence, have equal spectral multiplicities. Therefore, it suffices to prove Proposition 4.2 for $p = 2$.

Since the function $q(x)$ is not equivalent to a function $d_\pi(x)$ (or $-d_\pi(x)$) of the form (2.9) for any finite partition π of the form (2.8), the same is also true for the function $q(1-x)$. Thus, by virtue of Corollary 4.2, for any $n \in \mathbb{N}$, one can find a subspace $E_n \in \mathrm{Lat} V_{q(1-x),1}$ such that the operator $V_{q(1-x),1} \upharpoonright_{E_n}$ is quasisimilar to the operator $R := \bigoplus_{i=1}^{n} J_{m_i M_i 0} = \bigoplus_{i=1}^{n} J_{m_i 0} \oplus J_{0 M_i}$ acting in the space $\bigoplus_{i=1}^{n} (L_p[m_i,0] \oplus L_p[0,M_i])$. By virtue of Lemma 4.1, the operator R is quasisimilar to the operator $V_- \oplus V_+ := (\bigoplus_{i=1}^{n} m_i J) \oplus (\bigoplus_{i=1}^{n} M_i J)$ acting in $\bigoplus_{i=1}^{n} L_p[m_i,0] \bigoplus_{i=1}^{n} L_p[0,M_i]$. Since the numbers m_i and M_i cannot be simultaneously equal to zero, one can find either $[n/2]$ nonzero numbers $m_i < 0$ or $[n/2]$ nonzero numbers $M_i > 0$. Therefore, according to Proposition 3.2, we have either $\mu_{V_-} \geqslant [n/2]$ or $\mu_{V_+} \geqslant [n/2]$ and, hence, $\mu_R = \mu_{V_-\oplus V_+} \ge \max(\mu_{V_-}, \mu_{V_+}) \geqslant [n/2]$. Now let

$$K, L : L_2[0,1] \to L_2[0,1],$$

$$(Kf)(x) = q(1-x)f(1-x), \qquad (Lf)(x) = f(1-x).$$

Then $K^{-1}V_{q,1}K = V^*_{q(1-x),1}$ and $L^{-1}JL = J^*$ and, consequently, $\mu_{V_{q,1}} = \mu_{V^*_{q(1-x),1}}$ and $\mu_{R^*} = \mu_R$. Since the operator R^* is isomorphic to the quotient operator induced by the operator $V^*_{q(1-x),1}$ on E_n, we have $\mu_{V^*_{q(1-x),1}} \geqslant \mu_{R^*}$ and, hence,

$$\mu_{V_{q,1}} = \mu_{V^*_{q(1-x),1}} \geqslant \mu_{R^*} = \mu_R \geqslant [n/2].$$

Since n can be made arbitrarily large, we conclude that $\mu_{V_{q,1}} = \infty$. □

By combining Corollary 3.1 with Proposition 4.2, we arrive at the following criterion of cyclicity of the operator $V^\alpha_{q,1}$ for any function $q(x)$ ($|q(x)| \equiv 1$):

Corollary 4.3. *Let $|q(x)| \equiv 1$. Then the operator $V^\alpha_{q,1}$ is cyclic if and only if either $q \equiv 1$ or $q(x) = \chi_{[0,a]}(x) - \chi_{[a,1]}(x)$ for some $a \in [0,1]$ and α is odd.*

Example 4.1. (1) Let

$$c = \left(\int_0^1 \left|\sin\frac{1}{t}\right| dt\right)^{-1}, \quad q(x) = c\sin\frac{1}{x}, \quad R(x) := \int_0^x |q(t)|\, dt$$

$$\text{and} \quad r(x) := \operatorname{sign}(q(R^{-1}(x))).$$

Then, by virtue of Theorem 2.1, the operators $V_{q,1}$ and $V_{r,1}$ are quasisimilar and, hence, have equal spectral multiplicities. Since the function $sign(q(x))$ is not equivalent to a function $d_\pi(x)$ (or $-d_\pi(x)$) of the form (2.9) for any finite partition π of the form (2.8), the same is true for the function $r(x)$. Therefore, the operator $V_{r,1}$ and, hence, the operator $V_{q,1}$ have no finite-dimensional cyclic subspaces.

(2) Let $q(x) = c|\sin\frac{1}{x}|$. Then $r(x) := \operatorname{sign}(q(R^{-1}(x))) = 1$ for almost all $x \in [0,1]$. Therefore, the operators $V_{q,1}$ and J are quasisimilar and the operator $V_{q,1}$ is cyclic.

5. Criterion of Unicellularity of the Operator $V^\alpha_{q,w}$

The next proposition is a special case of one of the results established in [12, 13]. For the sake of completeness, we present its elementary proof for $p = 2$.

Proposition 5.1. *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and let $(b-a)^\alpha(d-c)^\alpha\lambda_1\lambda_2 < 0$. Then the equations*

$$\lambda_1 J_{ab}^\alpha X = X\lambda_2 J_{cd}^\alpha, \qquad X : L_p[c,d] \to L_p[a,b], \tag{5.1}$$

$$Y\lambda_1 J_{ab}^\alpha = \lambda_2 J_{cd}^\alpha Y, \qquad Y : L_p[a,b] \to L_p[c,d], \tag{5.2}$$

have only trivial solutions X and Y.

Proof. We prove (5.1). By Lemma 4.1, we can write

$$J_{ab}^\alpha = T_{ab}^{-1}(b-a)^\alpha J^\alpha T_{ab}, \qquad J_{cd}^\alpha = T_{cd}^{-1}(d-c)^\alpha J^\alpha T_{cd}.$$

This enables us to rewrite relation (5.1) in the form

$$\lambda_1 T_{ab}^{-1}(b-a)^\alpha J^\alpha T_{ab} X = X\lambda_2 T_{cd}^{-1}(d-c)^\alpha J^\alpha T_{cd}$$

or $\lambda_1(b-a)^\alpha J^\alpha T_{ab} X T_{cd}^{-1} = T_{ab} X T_{cd}^{-1}\lambda_2(d-c)^\alpha J^\alpha$. Therefore, to prove (5.1), it suffices to consider the case where $\lambda_1\lambda_2 < 0$, $a = c = 0$, and $b = d = 1$, i.e.,

$$\lambda_1 J^\alpha X = X\lambda_2 J^\alpha, \qquad J, X : L_p[0,1] \to L_p[0,1]. \tag{5.3}$$

For the sake of simplicity, we set $p = 2$. Let us show that $X = 0$. Relation (5.3) implies that $(\mathbb{I} - \lambda\lambda_1 J^\alpha)^{-1}X = X(\mathbb{I} - \lambda\lambda_2 J^\alpha)^{-1}$. Therefore, for arbitrary vectors $f \in L_2[0,1]$ and $g \in L_2[0,1]$, we obtain

$$\begin{aligned} 0 = \int_0^1 \lambda_1 \int_0^x & {E_\alpha}^{\lambda\lambda_1(x-t)}(Xf)(t)\,dt\,g(x)\,dx \\ & - \int_0^1 \lambda_2 \int_0^x {E_\alpha}^{\lambda\lambda_2(x-t)} f(t)\,dt\,(X^*g)(x)\,dx \\ & = \int_0^1 {E_\alpha}^{\lambda\lambda_1 t} \int_t^1 \lambda_1(Xf)(x-t)g(x)\,dx\,dt \\ & \qquad - \int_0^1 {E_\alpha}^{\lambda\lambda_2 t} \int_t^1 \lambda_2 f(x-t)(X^*g)(x)\,dx\,dt. \end{aligned}$$

By virtue of Lemma 3.2, $\int_t^1 \lambda_1(Xf)(x-t)g(x)\,dx\,dt = 0$ for almost all $t \in [0,1]$. We now set $g(x) = 1$ for almost all $x \in [0,1]$ and conclude that $(Xf)(x) = 0$ for almost all $x \in [0,1]$. Since the vector f is arbitrary, this means that $X = \mathbb{O}$.

For Eq. (5.2), the proof is similar. ∎

Definition 5.1. *An operator $T \in [X]$ is called unicellular if the lattice of its invariant subspaces $\operatorname{Lat} T$ is linearly ordered by inclusion, i.e., if $E_1, E_2 \in \operatorname{Lat} T$, then either $E_1 \subset E_2$ or $E_2 \subset E_1$.*

Theorem 5.1. *The following conditions are equivalent:*

1*a*. $\operatorname{Lat} V^\alpha_{q,w} = \operatorname{Lat} J^\alpha = \{\chi_{[a,1]} L_p[0,1] : 0 \leqslant a \leqslant 1\}$;

1*b*. $\operatorname{Cyc} V^\alpha_{q,w} = \operatorname{Cyc} J^\alpha$;

2. *the operator $V^\alpha_{q,w}$ is unicellular;*

3*a*. *for $x \in [0,1]$, the function $q(x)w(x)$ preserves its sign (almost everywhere);*

3*b*. *the function $Q(x) := \int_0^x q(t)w(t)\,dt$ is strictly monotone;*

3*c*. $E_q := \operatorname{span}\{q(x)Q(x)^n : n \geqslant 1\} = L_p[0,1]$;

4. *the operator $V^\alpha_{q,w}$ is quasisimilar to cJ^α, $c = \overline{c} \neq 0$.*

Proof. Since, for any two bounded operators A and B, the inclusion $\operatorname{Lat} A \subset \operatorname{Lat} B$ yields the inclusion $\operatorname{Cyc} B \subset \operatorname{Cyc} A$, the implication 1a $\Rightarrow$ 1b is true. Conversely, if assertion 1b is true, then, by virtue of Corollaries 3.1 and 4.3, assertion 1a is also true. The following equivalence relations are evident: $3a \Leftrightarrow 3b \Leftrightarrow 3c$ and $1 \Leftrightarrow 2$. By Corollary 3.1, we have 3a $\Leftrightarrow$ 1b and, by Theorem 2.1, 3a $\Rightarrow$ 4. We now prove that $4 \Rightarrow 3a$. Since the operator cJ^α is cyclic and the operator $V^\alpha_{q,w}$ is quasisimilar to the operator cJ^α, the operator $V^\alpha_{q,w}$ is also cyclic. By Theorem 2.1, we can restrict ourselves to the case where $|q(x)| = 1$ and $w(x) \equiv 1$. Then, according to Corollary 4.3, $q(x) \equiv 1$ for even α, which proves the theorem. If α is odd, then there exists a point $a \in (0,1)$ such that $\pm q(x) = \chi_{[0,a]}(x) - \chi_{[a,1]}(x)$. Clearly, it suffices to consider the case $q(x) = \chi_{[0,a]}(x) - \chi_{[a,1]}(x)$. Let X and Y be deformations connecting the operators cJ^α and $V^\alpha_{q,1}$, i.e., $cJ^\alpha Y = = Y V^\alpha_{q,1}$ and $X cJ^\alpha = V^\alpha_{q,1} X$. We now consider the block-matrix representation of the operators Y, cJ^α, and $V^\alpha_{q,1}$ relative to the decomposition of the space $L_p[0,1]$ into the direct sum $L_p[0,1] = L_p[0,a] \dotplus L_p[a,1]$:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \qquad cJ^\alpha = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \qquad V^\alpha_{q,w} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where $J_{11} = cJ_{0a}$, $J_{22} = cJ_{a1}$, $V_{11} = J_{0a}$, and $V_{22} = -J_{a1}$.

Since $L_p[a,1] \in \operatorname{Lat} V^\alpha_{q,1} \bigcap \operatorname{Lat} cJ$, we have $J_{12} = V_{12} = \mathbb{O}$.

Further, since $cJY = YV^\alpha_{q,w}$ and $XcJ = V^\alpha_{q,w}X$, we can write

$$J_{11}Y_{12} = Y_{12}V_{22}, \qquad J_{21}Y_{12} + J_{22}Y_{22} = Y_{22}V_{22}$$

$$X_{12}J_{22} = V_{11}X_{12}, \qquad X_{11}J_{11} + X_{12}J_{21} = V_{11}X_{11}$$

or

$$cJ_{0a}Y_{12} = -Y_{12}J_{a1}, \qquad J_{21}Y_{12} + cJ_{a1}Y_{22} = -Y_{22}J_{a1},$$

$$X_{12}cJ_{a1} = J_{0a}X_{12}, \qquad X_{11}cJa1 + X_{12}J_{21} = J_{a1}X_{11}.$$

If $c > 0$, then, by virtue of Proposition 5.1, $Y_{12} = \mathbb{O}$ and $Y_{22} = \mathbb{O}$, i.e., the kernel of the operator Y is nonempty. If $c < 0$, then, according to Proposition 5.1, $X_{12} = \mathbb{O}$ and $X_{11} = \mathbb{O}$, i.e., the image of the operator X is not dense. Thus, in order that the operators cJ and $V^{\alpha}_{q,w}$ be quasisimilar, it is necessary that the function q be sign preserving. □

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